

PARAMETRIC ESTIMATION OF EXTINCTION PROBABILITY OF BRANCHING PROCESS BY USING JOINT ESTIMATION OF MEAN AND VARIANCE

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ABSTRACT

The paper deals with the joint asymptotic normality of estimators of offspring mean and the offspring variance of the Bienayme-Galton-Watson Branching process, when n generation sizes are observable. Using these results, the probability of ultimate extinction is also estimated in the parametric set up. As an illustration, an application of the method to the case of fractional linear generating function is demonstrated. The method is also applied for H1N1 positive cases data generated during 2009.

Key words and Phrases: *Bienayme-Galton-Watson branching process, Joint asymptotic normality, Martingale limit theory.*

1 Introduction

Bienayme-Galton-Watson(BGW)branching process is being extensively used to model various situations and phenomenon in epidemiology, nuclear science, life science. The estimation of various parameters of this process is studied in detail in

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the literature[c.f. Pakes(1975), Becker(1977)]. The estimation procedures are developed for parameters such as offspring mean(mean of the offspring distribution) also referred as growth as well as criticality parameter[c.f. Heyde and Seneta(1972), Dion(1974)], the estimation of offspring variance [c.f. Heyde(1974), Dion(1975)]and for the another parameter of interest the extinction probability of the process. The estimation of the extinction probability is studied via estimation of offspring distribution and probability generating function(p.g.f.) of offspring distribution.

The above mentioned exercises are carried out under different observational schemes. The nonparametric estimation of offspring mean and offspring variance can be carried out when only n generation sizes are observed for the process(assuming that the process does not become extinct before n^{th} generation). However, to carry out estimation of offspring distribution, offspring p.g.f. and finally extinction probability non parametrically one needs to observe individual offspring sizes. In practice, it is not always possible to observe individual offspring sizes. For example, the spread of HIV virus can be measured by observing the viral load only, another example is that of H1N1 wherein it is impossible to note how many individuals have got the infection from one particular individual. Hence, one needs to study estimation of these parameters using parametric set up that is assuming certain form for the offspring distribution.

All the estimation procedures mentioned above deal with the estimation of parameters separately and very little has been done to make joint estimation of mean and variance of offspring distribution. Godambe and Heyde(1987) have studied joint estimation of these parameters when n generation sizes are observable, using estimating function approach proposed by Godambe(1985). They have called these estimators as O_F optimal. The present article attempts to provide an analogous result in asymptotic set up and it is shown that these estimators which are jointly O_F optimal are also O_A optimal a term coined by Godambe and Heyde(1987)for best asymptotically normal estimator/s. The point to be noted is that these exercises do not assume form for the offspring distribution. However when one wants to go to O_A optimality for estimators of offspring p.g.f. and extinction probability using

the estimators of mean and variance one needs to assume the form for the offspring p.g.f.

The paper is organised as follows, the first section is introduction whereas section two discusses joint estimation of offspring mean and offspring variance. The asymptotic properties of the estimators are established using martingale limit theory and Cramer-Wold Principle. The third section is devoted to the estimation of extinction probability when the offspring distribution is assumed to be fractional linear generating function. Again asymptotic properties of the estimator of extinction probability is also discussed. Last section that is, section four includes a case study for H1N1 spread or Swine flu epidemic for Pune 2009 data.

The next section discusses the joint estimation of offspring mean and offspring variance.

2 Estimation of offspring parameters

The Bienayme-Galton-Watson(BGW) Branching process $\{Z_n, n = 0, 1, 2, \dots\}$ is defined by recursive formula satisfying following relation,

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i(n), \quad i = 0, 1, 2, \dots; n = 1, 2, \dots$$

Where

i) $Z_0 = 1$ a.s.

ii) $\{\xi_i(n), i = 1, 2, \dots; n = 1, 2, \dots\}$ are independent and identically distributed (i.i.d.) random variables having distribution

$$\{p_k, k = 0, 1, 2, \dots\}, 0 < p_k < 1, \sum_{k=0}^{\infty} p_k = 1$$

The distribution of Z_1 which is same as that of $\xi_1(1)$ is called as offspring distribution. Further let, $m = E[Z_1]$ and $\sigma^2 = Var(Z_1)$ are called as offspring mean and offspring variance respectively. It assume that $m < \infty$ and $0 < \sigma^2 < \infty$. As mentioned earlier, m is growth as well as criticality parameter.

The aim of this section is to study the joint estimation of m and σ^2 , when n generation sizes of the process $Z_1, Z_2, \dots, Z_n$ are observed. Note that, it is assumed here

that the process does not become extinct before the n^{th} generation. An efficient estimator of mean in the literature is given by,

$$\hat{m}_n = \left(\sum_{i=1}^n Z_i \right) \left(\sum_{i=1}^n Z_{i-1} \right)^{-1}$$

and the estimator of the variance is given by

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^n (Z_i - \hat{m}_n Z_{i-1})^2 Z_{i-1}^{-1}$$

The $\hat{m}_n$ is called as Harris estimator where as estimator σ_n^2 was proposed and studied by Heyde(1974). It is also established that these estimators are CAN and are hence O_A optimal separately for m and σ^2 . As reported in Godambe(1985) these are also finite sample optimal separately, Godambe also has demonstrated that these are jointly finite sample optimal when μ_3 , the third central moment of offspring distribution that is $E[Z_1 - m]^3 = 0$. Hence, $\hat{m}_n$ and $\hat{\sigma}_n^2$ are separately O_A as well as O_F optimal. However, they are jointly O_F optimal under the condition $\mu_3 = 0$.

Now, we proceed to study the joint optimality in asymptotic sense that is O_A optimality. Suppose as mentioned earlier data are $Z_1, \dots, Z_n$, n generation sizes of the process. The goal is to establish CAN property of $(\hat{m}_n, \hat{\sigma}_n^2)$. The asymptotic properties of $\hat{\sigma}_n^2$ is studied by studying first the asymptotic properties of

$$\tilde{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^n (Z_i - m Z_{i-1})^2 Z_{i-1}^{-1}$$

in-view of the following lemma

Lemma 2.1 :- In the supercritical case on the non-extinction path with $0 < \sigma^2 < \infty$ and $\epsilon > 0$; $n^{1-\epsilon}(\tilde{\sigma}_n^2 - \hat{\sigma}_n^2)$ converges in probability to zero.

Proof:- Consider,

$$n^{1-\epsilon}(\tilde{\sigma}_n^2 - \hat{\sigma}_n^2) = \frac{1}{n} \sum_{i=1}^n Z_{i-1}(\hat{m}_n - m)^2 n^{-\epsilon}$$

By using the asymptotic Normality of estimator of offspring mean that is

$\sigma^{-1} \sqrt{\sum_{i=1}^n Z_{i-1}(\hat{m}_n - m)^2}$ has Standard Normal distribution, it can be concluded that, $\sigma^{-2} \sum_{i=1}^n Z_{i-1}(\hat{m}_n - m)^2$ has asymptotically chi-square distribution with one degree of freedom.

As $n^{-\epsilon} \rightarrow 0$ for $\epsilon > 0$ as $n \rightarrow \infty$ $n^{1-\epsilon}(\tilde{\sigma}_n^2 - \hat{\sigma}_n^2) \xrightarrow{p} 0$

Hence the proof.

The following theorem establishes the CAN property of $(\hat{m}_n, \hat{\sigma}_n^2)$ that is joint asymptotic properties known as O_A optimality.

Theorem 2.1:- In the supercritical case on a non extinction path with $0 < E(Z_1)^4 < \infty$ as $n \rightarrow \infty$,

- i) $(\hat{m}_n, \tilde{\sigma}_n^2)$ jointly consistent for (m, σ^2)
 ii) $\Sigma_n^{-1/2} \begin{bmatrix} \sum_{k=1}^n Z_{k-1}(\hat{m}_n - m) \\ n(\tilde{\sigma}_n^2 - \sigma^2) \end{bmatrix} \rightarrow N_2 \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$

where,

$$\Sigma_n = \begin{bmatrix} \sigma^2 \sum_{k=1}^n Z_{k-1} & n\mu_3 \\ n\mu_3 & 2n\sigma^4 + (\mu_4 - 3\sigma^4) \sum_{i=1}^n Z_{k-1}^{-1} \end{bmatrix}$$

and $\Sigma_n^{-1/2}$ is inverse of the matrix $\Sigma_n^{1/2}$, which is obtained by spectral decomposition $\Sigma_n^{1/2} = \Gamma \Lambda^{1/2} \Gamma$ where, Λ is a diagonal matrix of eigen values and Γ consists of corresponding eigen vectors of the matrix Σ_n

Proof:-

- i) Since $\hat{m}_n$ is consistent estimator of m and $\tilde{\sigma}_n^2$ is consistent for σ^2

The proof follows by usual result of joint consistency [c.f. Rao(1968)].

- ii) Consider the vector

$$\begin{pmatrix} \sum_{i=1}^n (Z_{i-1})(\tilde{m}_n - m) \\ n(\tilde{\sigma}_n^2 - \sigma^2) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n (Z_i - mZ_{i-1}) \\ \sum_{i=1}^n (Z_{i-1}^{-1}(Z_i - mZ_{i-1})^2 - \sigma^2) \end{pmatrix}$$

Consider,

$$\begin{aligned} l_1 \sum_{i=1}^n Z_{i-1}(\hat{m}_n - m) + l_2 n(\hat{\sigma}_n^2 - \sigma^2) &= l_1 \sum_{i=1}^n (Z_i - mZ_{i-1}) + l_2 \sum_{i=1}^n (Z_{i-1}^{-1}(Z_i - mZ_{i-1})^2 - \sigma^2) \\ &= \sum_{i=1}^n U_i \\ &= X_n \end{aligned} \tag{2.1}$$

Since,

$$E[(Z_i - mZ_{i-1})|Z_1, Z_2, \dots, Z_{i-1}] = 0$$

$$E[(Z_{i-1}^{-1}(Z_i - mZ_{i-1})^2 - \sigma^2)|Z_1, Z_2, \dots, Z_{i-1}] = 0,$$

X_n is a zero mean Martingale with respect to $\mathcal{F}_n = \sigma\{Z_1, Z_2, \dots, Z_{i-1}\}$

$$V_n^2 = \sum_{i=1}^n E[U_i^2|\mathcal{F}_{i-1}]$$

Now consider,

$$E[U_i^2|\mathcal{F}_{i-1}]$$

$$= E\{[l_1(Z_i - mZ_{i-1}) + l_2(Z_{i-1}^{-1}(Z_i - mZ_{i-1})^2 - \sigma^2)]^2|\mathcal{F}_{i-1}\}$$

$$= l_1^2\sigma^2 Z_{i-1} + 2l_1 l_2 \mu_3 + l_2^2\sigma^4 - 2l_2^2\sigma^4 + l_2^2 Z_{i-1}^{-2}(3\sigma^2 Z_{i-1}^2 - 3Z_{i-1}\sigma^4 + Z_{i-1}\mu_4)$$

$$= l_1^2\sigma^2 Z_{i-1} + 2l_1 l_2 \mu_3 + 2l_2^2\sigma^4 + l_2^2(Z_{i-1}^{-1})(\mu_4 - 3\sigma^4)$$

Thus,

$$V_n^2 = l_1^2\sigma^2 \sum_{i=1}^n Z_{i-1} + 2l_1 l_2 n\mu_3 + 2l_2^2\sigma^4 n + l_2^2 \sum_{i=1}^n Z_{i-1}^{-1}(\mu_4 - 3\sigma^4)$$

$$s_n^2 = l_1^2\sigma^2 \sum_{i=1}^n E(Z_{i-1}) + 2l_1 l_2 n\mu_3 + 2l_2^2\sigma^4 n + l_2^2 \sum_{i=1}^n E(Z_{i-1}^{-1})(\mu_4 - 3\sigma^4)$$

Since, $\{Z_n m^{-n}\}$ is a martingale with respect to $\mathcal{F}_n = \sigma - \{Z_1, \dots, Z_n\}$, by martingale convergence theorem[c.f. Hall and Heyde(1980)] it converges almost surely(a.s.) to $W > 0$ as $n \rightarrow \infty$ on non-extinction path. Also note that in the supercritical case on non-extinction path $P[W > 0] = 1$ [c.f. Athreya and Ney(1972),Guttorp(1991)]. Further, using this fact and Toeplitz's lemma one gets,

$$V_n^2/s_n^2 \rightarrow W \quad a.s. \quad (2.2)$$

Using similar argument

$$s_n^2 s_{n-r}^{-2} \rightarrow m^{-r}, m > 1, r > 0 \quad (2.3)$$

Now to prove third condition one has to show that,

$$E[\exp(\frac{itU_n}{aV_n})|\mathcal{F}_n] \rightarrow \exp(\frac{-t^2}{2})$$

Note that,

$$\begin{aligned} & \frac{U_n}{aV_n} \\ &= \frac{l_1(Z_n - mZ_{n-1}) + l_2(Z_{n-1}^{-1}(Z_n - mZ_{n-1})^2 - \sigma^2)}{aV_n} \\ &= \frac{\sum_{i=1}^{Z_{n-1}} (\xi_{n-1} - m)}{l_1 \sigma \sqrt{Z_{n-1}}} \frac{\sigma \sqrt{Z_{n-1}}}{aV_n} + \frac{l_2(Z_{n-1}^{-1}(Z_n - mZ_{n-1})^2 - \sigma^2)}{aV_n} \end{aligned}$$

Using Random sum Central limit theorem

$$\frac{\sum_{i=1}^{Z_{n-1}} (\xi_{n-1} - m)}{l_1 \sigma \sqrt{Z_{n-1}}} \rightarrow N(0, 1) \quad \text{as } n \rightarrow \infty$$

and the second term vanishes since $(Z_{n-1}^{-1}(Z_n - mZ_{n-1})^2 - \sigma^2) \xrightarrow{P} 0$. [c.f. Dion (1975), Heyde(1974)]

$$V_n^{-1} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

Hence,

$$\frac{l_1 \sigma \sqrt{Z_{n-1}}}{aV_n} \rightarrow (\frac{m}{m-1})^{1/2} = a$$

$$\frac{U_n}{aV_n} \rightarrow N(0, 1)$$

$$E \left[\exp(\frac{itU_n}{aV_n} | \mathcal{F}_n) \right] \rightarrow \exp(\frac{-t^2}{2}) \quad (2.4)$$

In view of the above one can conclude that the conditions of Scott's central limit theorem for martingale are satisfied [c.f. Scott(1978)].

We have,

$$\begin{aligned} X_n &= l_1 \sum_{k=1}^n (Z_k - mZ_{k-1}) + l_2 \sum_{k=1}^n ((Z_{k-1}^{-1})(Z_k - mZ_{k-1})^2 - n\sigma^2) \\ &= \sum_{k=1}^n U_k \end{aligned}$$

X_n is a zero mean martingale with respect to $\mathcal{F}_n = \sigma\{Z_1, Z_2, \dots, Z_{k-1}\}$

It is already shown in (2.2)

$$V_n^2/s_n^2 \xrightarrow{P} W = \eta^2(\text{say}) \text{ a.s. as } n \rightarrow \infty, E(\eta^2) < \infty$$

Also by equation (2.3) ,

$$s_n^{-2} s_{n-r}^2 \rightarrow m^{-r}, m > 1, r > 0$$

In view of equation (2.4) that

$$E[e^{itX_n/aV_n} | \mathcal{F}_n] \xrightarrow{P} \exp(-t^2/2)$$

Hence, all the three conditions are satisfied.

To apply Scott's Central limit theorem(CLT) for martingale, Let us define,

$$\xi_{nk} = \begin{cases} \frac{U_{n-k}}{as_n}, & k = 0, 1, 2, \dots, n \\ 0, & k > n \end{cases}$$

Here, let

$$\xi_{nk} = \frac{l_1 \sum_{k=1}^n (Z_{n-k} - mZ_{n-k+1}) + l_2 \sum_{k=1}^n (Z_{n-k+1}^{-1})(Z_k - m\phi(Z_{n-k+1}))^2 - (n-k)\sigma^2}{as_n}$$

further let

$\xi = (\eta Y_0, \eta Y_1, \dots)$ Where, $\eta \geq 0$ is almost surely positive random variable and is distributed independently of $Y_0, Y_1, \dots$, which are also mutually independent with Y_k

distributed as $N(0, m^{-k})$ Now consider,

$$\begin{aligned} & V_n^{-1} [l_1 \sum_{k=1}^n (Z_k - mZ_{k-1}) + l_2 \sum_{k=1}^n ((Z_{k-1}^{-1})(Z_k - m\phi(Z_{k-1}))^2 - \sigma^2)] \\ &= V_n^{-1} as_n \sum_{k=1}^n \xi_{nk} \\ &= a \sum_{k=1}^{\infty} \xi_{nk}^* \quad \xi_{nk}^* = 0 \text{ for } k \geq n \end{aligned}$$

$$= a \limsup \sum_{k=1}^{n-1} \xi_{nk}^* \\ = ah(\xi_{n0}^*, \xi_{n1}^*, \dots)$$

Where, $h(x_0, \dots) = \limsup \sum_{k=1}^{n-1} x_k$ is a continuous function

Let, $\xi_n^* = (\xi_{n0}^*, \xi_{n1}^*, \dots) = s_n \xi_{nk} V_n^{-1}$ and $\xi^* = (Y_0, Y_1, \dots)$

where, $Y_0, Y_1, \dots$ are independent r.v. with $N(0, m^{-k})$

By Scott CLT $\xi_n^* \rightarrow \xi^*$ in distribution as $n \rightarrow \infty$ and by continuous mapping theorem [c.f Billingsley(1968)], $h(\xi_n^*) \xrightarrow{L} h(\xi^*)$

Note that, $\sum_{k=1}^n Y_k$ is a zero mean martingale.

Further note that,

$$E[|\sum_{k=1}^n Y_k|] \leq E[(\sum_{k=1}^n Y_k)^2]^{1/2} = [\sum_{k=1}^n E[Y_k^2] \sum_k \sum_j E[Y_k Y_j]]^{1/2} < \infty$$

By martingale convergence theorem [c.f. Hall and Heyde(1980)] $\sum_{k=1}^n Y_k$ converges

almost surely and hence in law to Y . Further $\sum_{k=1}^{n-1} Y_k$ has normal distribution with

mean zero and variance $\sum_{k=1}^{n-1} m^{-k} = \frac{1-m^{-1}}{1-m} = \beta_n^2$, which converges to $(1-m^{-1})^{-1} =$

a^{-2} . For any real x , $P[\sum_{k=1}^{n-1} Y_k \leq x] = [x\beta_n^{-1}]$

where, ϕ denotes the distribution function of standard normal. By continuity of ϕ ,

β_n^{-1} converges to $\phi(xa)$. Therefore,

$$\sum_{k=1}^{n-1} Y_k \xrightarrow{L} N(0, a^{-2})$$

$$a \sum_{k=1}^{n-1} Y_k \xrightarrow{L} N(0, 1)$$

Hence,

$$\frac{l_1(\sum_{k=1}^n Z_{k-1})(\hat{m}_n - m) + l_2 n(\tilde{\sigma}_n^2 - \sigma^2)}{V_n} \rightarrow N(0, 1) \quad (2.5)$$

Further as a consequence of (2.5) and Cramer Wold principle, one gets,

$$\Sigma_n^{-1/2} \begin{bmatrix} \sum_{k=1}^n Z_{k-1}(\hat{m}_n - m) \\ n(\tilde{\sigma}_n^2 - \sigma^2) \end{bmatrix} \rightarrow N_2 \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

where,

$$\Sigma_n = \begin{bmatrix} \sigma^2 \sum_{k=1}^n Z_{k-1} & n\mu_3 \\ n\mu_3 & 2n\sigma^4 + (\mu_4 - 3\sigma^4) \sum_{i=1}^n (Z_{k-1}^{-1}) \end{bmatrix}$$

Observe that, if $\mu_3 = 0$ the two estimators are asymptotically normal and uncorrelated and hence are independent. This fact is consistent with O_F optimality condition suggested by Godambe and Heyde(1987). Under same orthogonality type of condition these estimators are O_A as well as O_F optimal, when $\tilde{\sigma}_n$ is used.

The next theorem discusses the results for joint CAN property of $(\hat{m}_n, \hat{\sigma}_n)$.

Theorem 2.2:- In supercritical case on a non extinction path with $0 < E(Z_1)^4 < \infty$ as $n \rightarrow \infty$

i) $(\hat{m}_n, \hat{\sigma}_n^2)$ jointly consistent for (m, σ^2)

$$\text{ii) } \Sigma_n^{-1/2} \begin{bmatrix} \sum_{k=1}^n Z_{k-1}(\hat{m}_n - m) \\ n(\hat{\sigma}_n^2 - \sigma^2) \end{bmatrix} \rightarrow N_2 \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

where,

$$\Sigma_n = \begin{bmatrix} \sigma^2 \sum_{k=1}^n Z_{k-1} & n\mu_3 \\ n\mu_3 & 2n\sigma^4 + (\mu_4 - 3\sigma^4) \sum_{i=1}^n Z_{k-1}^{-1} \end{bmatrix}$$

Proof:-

The result (i) is consequence of theorem 2.1 and lemma 2.1. To prove result (ii) consider,

$$\begin{bmatrix} \sum_{k=1}^n Z_{k-1}(\hat{m}_n - m) \\ n(\hat{\sigma}_n^2 - \sigma^2) \end{bmatrix} - \begin{bmatrix} \sum_{k=1}^n Z_{k-1}(\hat{m}_n - m) \\ n(\tilde{\sigma}_n^2 - \sigma^2) \end{bmatrix} = \begin{bmatrix} 0 \\ n^\epsilon n^{1-\epsilon}(\tilde{\sigma}_n^2 - \hat{\sigma}_n^2) \end{bmatrix}$$

From lemma 2.1, in the supercritical case on the non-extinction path with $0 < \sigma^2 < \infty$ and $\epsilon > 0$; $n^{1-\epsilon}(\tilde{\sigma}_n^2 - \hat{\sigma}_n^2)$ converges in probability to zero a.s. and $n^\epsilon \rightarrow 0$ as $n \rightarrow \infty$

$$\text{Hence, } \begin{bmatrix} 0 \\ n(\tilde{\sigma}_n^2 - \hat{\sigma}_n^2) \end{bmatrix} \xrightarrow{P} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ as } n \rightarrow \infty$$

Therefore,
$$\begin{bmatrix} \sum_{k=1}^n Z_{k-1}(\hat{m}_n - m) \\ n(\hat{\sigma}_n^2 - \sigma^2) \end{bmatrix} - \begin{bmatrix} \sum_{k=1}^n Z_{k-1}(\hat{m}_n - m) \\ n(\tilde{\sigma}_n^2 - \sigma^2) \end{bmatrix} \xrightarrow{P} 0$$

Therefore,

$$\Sigma_n^{-1/2} \begin{bmatrix} \sum_{k=1}^n Z_{k-1}(\hat{m}_n - m) \\ n(\hat{\sigma}_n^2 - \sigma^2) \end{bmatrix} \rightarrow N_2 \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

where,

$$\Sigma_n = \begin{bmatrix} \sigma^2 \sum_{k=1}^n Z_{k-1} & n\mu_3 \\ n\mu_3 & 2n\sigma^4 + (\mu_4 - 3\sigma^4) \sum_{i=1}^n (Z_{k-1}^{-1}) \end{bmatrix}$$

Observe that, when $\mu_3 = 0$ the two estimators are asymptotically normal and uncorrelated and hence are independent. This fact is consistent with O_F optimality condition suggested by Godambe and Heyde(1987). Note that, O_A and O_F optimality can be achieved under same condition.

3 Estimation of mean and variance in case of Fractional linear generating function

The present section deals with the estimation of offspring probability by using results on joint estimation of offspring mean and variance derived in the last section. Assume that, the offspring p.g.f. is in the fractional linear generating function form which is given by,

$$f(s) = 1 - \frac{b}{1-c} + \frac{bs}{1-cs} \quad (3.1)$$

The mean of this distribution,

$$m = \frac{b}{(1-c)^2} \quad (3.2)$$

And the variance is given by,

$$\sigma^2 = \frac{b(1-c^2-b)}{(1-c)^4} \quad (3.3)$$

The parameters b and c can be estimated by

$$\hat{b} = \frac{4\hat{m}_n^3}{(\hat{\sigma}_n^2 + \hat{m}_n^2 + \hat{m}_n)^2}$$

$$\hat{c} = \frac{\hat{\sigma}_n^2 + \hat{m}_n^2 - \hat{m}_n}{\hat{\sigma}_n^2 + \hat{m}_n^2 + \hat{m}_n}$$

The form of extinction probability for the BGW process with fractional linear generating function as offspring pgf is given by Stigler(1970) which is as follows

$$\pi = \min\{1, \frac{1-b-c}{c(1-c)}\}$$

By using above estimators of b and c the extinction probability can be estimated by

$$\hat{\pi} = \min\{1, \frac{\hat{\sigma}_n^2 - \hat{m}_n^2 + \hat{m}_n}{\hat{\sigma}_n^2 + \hat{m}_n^2 - \hat{m}_n}\}$$

We now proceed to prove the CAN property of $\hat{\pi}$ by using joint O_A optimality of $\hat{m}_n^2$ and $\hat{\sigma}_n^2$.

Theorem 3.1:- The estimator for extinction probability $\hat{\pi}_n$ is consistent and asymptotically normal for π .

Proof:- The extinction probability is the function of $\hat{m}_n^2$ and $\hat{\sigma}_n^2$. By using theorem 2.1, and noting that π is a function of b and c and hence that of m and σ^2 with derivative matrix or matrix of transformation. $D = \begin{bmatrix} \frac{d\pi}{d\hat{m}_n} \\ \frac{d\pi}{d\hat{\sigma}_n^2} \end{bmatrix}$

Using standard results in asymptotic inference theory, one can get CAN property of estimator of extinction probability. The estimator for extinction probability $\hat{\pi}_n$ is consistent and asymptotically normal for π that is

$$(D'\Sigma_n D)^{-1/2}(\hat{\pi}_n - \pi) \xrightarrow{D} N(0, 1)$$

where,

$$D' = \begin{bmatrix} \frac{d\pi}{d\hat{m}_n} & \frac{d\pi}{d\hat{\sigma}_n^2} \end{bmatrix}$$

and

$$\Sigma_n = \begin{bmatrix} \sigma^2 \sum_{k=1}^n Z_{k-1} & n\mu_3 \\ n\mu_3 & 2n\sigma^4 + (\mu_4 - 3\sigma^4) \sum_{k=1}^n Z_{k-1}^{-1} \end{bmatrix}$$

4 Case study(H1N1)

This section is devoted to discuss an application of derived results to H1N1 data. The city of Pune in India was affected by this epidemic started in July 2009. The virus responsible for spreading this epidemic is named as H1N1 or Swine flu virus. The data used for the study is a secondary data, Which has number of positive cases observed per day during 15th July 2009 to 30th September 2009 in Pune city. Let us derive estimates of m and σ^2 and the extinction probability for this epidemic by modelling its spread as simple BGW process.

Since, the data has number of positive cases per day (for 77 days) we can consider that as 77 generations of the BGW branching process. The estimates for mean and variance of the offspring distribution are,

$$\hat{m}_n = \left(\sum_{i=1}^n Z_i \right) \left(\sum_{i=1}^n Z_{i-1} \right)^{-1} = 1.004909984$$

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^n (Z_i - \hat{m}_n Z_{i-1})^2 Z_{i-1}^{-1} = 0.370476787$$

Therefore the estimates of parameters of Fractional linear generating function b and c are,

$$\hat{b} = \frac{4\hat{m}_n^3}{(\hat{\sigma}_n^2 + \hat{m}_n^2 + \hat{m}_n)^2} = 0.87650434$$

$$\hat{c} = \frac{(\hat{\sigma}_n^2 + \hat{m}_n^2 - \hat{m}_n)^2}{(\hat{\sigma}_n^2 + \hat{m}_n^2 + \hat{m}_n)^2} = 0.066071874$$

By using above estimators of b and c the extinction probability can be estimated by

$$\hat{\pi} = \frac{\hat{\sigma}_n^2 - \hat{m}_n^2 + \hat{m}_n}{\hat{\sigma}_n^2 + \hat{m}_n^2 - \hat{m}_n} = 0.930597218$$

The asymptotic property for $\hat{\pi}_n$ can be derived using the estimates derived above and the computed values of D matrix and Σ_n matrix to get,

$$(D' \Sigma_n D)^{-1/2} (\hat{\pi}_n - \pi) \rightarrow N(0, 1)$$

The estimate of extinction probability is 0.930597218 suggests that the epidemic is not going to die immediately. It gradually dies out with intensity in every season goes on reducing . The subsequent observations in the seasons 2011,2012,2013 the same fact was observed still there are cases of positive H1N1 sample, however the

number is decreasing from season to season. The above fact amply justifies the merit of the model.

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Appendix 1 The Scott(1978) central limit for martingales is stated as follows
Let $\{X_n, \mathcal{F}_n; n \geq 0\}$ be a martingale defined on a probability space (Ω, A, P) with

$$X_n = \sum_{k=0}^n U_k, \quad s_n^2 = E(X_n)^2 < \infty$$

$$V_n^2 = \sum_{k=0}^n E(U_k^2 | \mathcal{F}_n) \text{ where } \mathcal{F}_{-1} = \{\phi, \Omega\}$$
For $p > 1$ and $a > 0$ define random sequence in $\mathcal{R}^N$, ξ and ξ_n for every $n \geq 1$ by,

$$\xi_n = \{\xi_{n0}, \xi_{n1}, \dots\},$$

$$\xi_{nk} = \begin{cases} U_{n-k}/as_n & k=0,1,2,\dots,n \\ 0 & k \neq n \end{cases}$$

and

$$\xi = (\eta Y_0, \eta Y_1, \dots)$$

Where, $\eta \geq 0$ and is distributed independently of $Y_0, Y_1, \dots$ which are also mutually independent with Y_k distributed as $N(0, p^{-k})$

Also define,

$$\xi_n^* = (\xi_{n0}^*, \xi_{n1}^*, \dots) = s_n V_n^{-1} \xi_n$$

and

$$\xi^* = (Y_0, Y_1, \dots)$$

Suppose that ,

$$s_n^{-2} V_n^2 \xrightarrow{P} \eta^2 \quad \text{with } E(\eta^2) < \infty$$

$$s_n^{-2} s_{n-r}^2 \xrightarrow{P} p^{-r} \quad \text{for } r = 0, 1, 2, \dots$$

and

$$E[e^{itU_n/aV_n}|\mathcal{F}_n] \xrightarrow{P} \exp(-t^2/2) \text{ on } \eta > 0$$

then for every $E \in \mathcal{D}$ such that,

$$P[\xi \in \partial\mathcal{E}] = 0 \text{ and ever } F \in \mathcal{F}$$

$$P[\{\xi_n \in \mathcal{E}, F|\eta > 0\}] \rightarrow P[\xi \in \mathcal{E}]P[F|\eta > 0]$$

and for every $E \in \mathcal{D}$ such that,

$$P[\xi^* \in \partial\mathcal{E}] = 0 \text{ and ever } F \in \mathcal{F}$$

$$P[\{\xi_n^* \in \mathcal{E}, F|\eta > 0\}] \rightarrow P[\xi^* \in \mathcal{E}]P[F|\eta > 0]$$

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