

L-MOMENTS AND TL - MOMENTS ESTIMATION AND RECURRENCE RELATIONS FOR HIGHER MOMENTS OF ORDER STATISTICS FROM LINDLEY DISTRIBUTION

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ABSTRACT

In this paper, we have obtained L-moments and TL-moments of Lindley distribution and used them to find the L-moments and TL-moments estimator of the parameter θ of the Lindley distribution. Also we have obtained the recurrence relations for higher moments of order statistics arising from doubly truncated Lindely distribution which enable one to obtain all the single, double (product) and higher moments of any order of all order statistics for any sample size in a simple recursive manner. Also, the corresponding recurrence relations for untruncated Lindley distribution have been obtained as special cases.

Keywords : Order Statistics, L-moments, TL-moments, Single moments, Product moments, Higher moments, Recurrence relations, Lindley distribution, Doubly truncated Lindley distribution.

1 Introduction

The Lindley distribution was first introduced in the literature by Lindley (1958) in connection with the Fiducial distribution and Bayes theorem. The probability

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density function (pdf) of Lindley distribution is given by (see Fig. 1)

$$f(x) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}, \quad x > 0, \quad \theta > 0 \quad (1)$$

and the cumulative distribution function (cdf) is given by

$$F(x) = 1 - \frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}, \quad x > 0, \quad \theta > 0. \quad (2)$$

Ghitany et al. (2008a) studied the statistical properties of Lindley distribution,

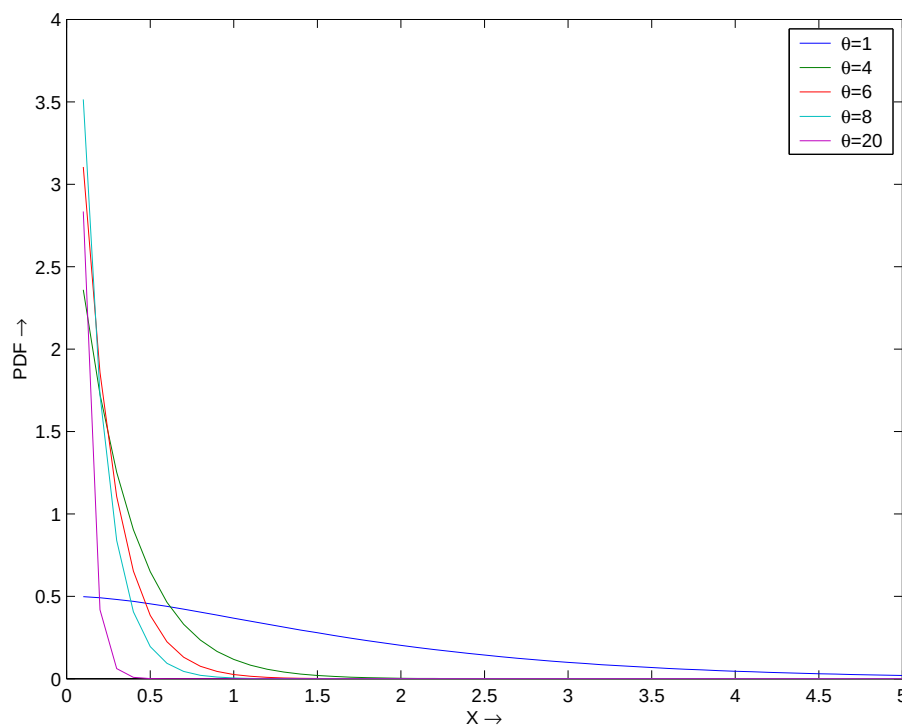


Figure 1: pdf of Lindley Distribution

showing this distribution may provide a better fitting than exponential distribution. The Lindley distribution is important for studying stress-strength reliability modelling (see Hussain (2006), Nadarajah et al. (2011)). For a more flexible model for life time data, Bakouch et al. (2012) introduced extended Lindley distribution. Some statisticians namely Sankaran (1970), Mahmoudi and Zakerzadeh (2010), Ghitany et al. (2008b) and Ghitany and Al-Mutairi (2009) have proposed new classes

of distributions based on modification of Lindley distribution.

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from the Lindley distribution given in (1), and let $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ be the corresponding order statistics, and $X_{r_1:n} \leq X_{r_2:n} \leq \dots \leq X_{r_k:n}$ be the corresponding sub-sample order statistics of size $k \leq n$. Let us denote the single moments $E(X_{r:n}^k)$ by $\mu_{r:n}^{(k)}$ ($1 \leq r \leq n$) and the product moments of k order statistics, viz. $E(X_{r_1:n}^{i_1} X_{r_2:n}^{i_2} \dots X_{r_k:n}^{i_k})$ by $\mu_{r_1, r_2, \dots, r_k:n}^{(i_1, i_2, \dots, i_k)}$ ($1 \leq r_1 < r_2 < \dots < r_k \leq n$ and $i_1, i_2, \dots, i_k = 0, 1, 2, \dots$).

One can observe from (1) and (2) that the characterizing differential equation for the case of Lindley distribution (untruncated case) is given by

$$(1 + \theta + \theta x)f(x) = \theta^2(1 + x)[1 - F(x)]. \quad (3)$$

In this paper we have derived L-moments and TL-moments of Lindley distribution and used them to obtain the L-moments and TL-moments estimator of the parameter θ of the Lindley distribution. We have also established several recurrence relations satisfied by the higher moments of order statistics from doubly truncated Lindley distribution as defined in Section 3. Further, the corresponding recurrence relations for the untruncated Lindley distribution, defined in (1), have been deduced as special cases.

2 L-moments and TL-moments

It may be mentioned that Hosking (1990) introduced the L-moments as a linear combination of probability weighted moments. Similar to ordinary moments, L-moments can also be used for parameter estimation, interval estimation and hypothesis testing. Hosking has shown that first four L-moments of a distribution measure, respectively, the average, dispersion, symmetry and tail weight (or peakedness) of the distribution. L-moments have turn out to be a popular tool in parametric estimation and distribution identification problems in different scientific areas such as hydrology in the estimation of flood frequency, etc. (see e.g. Hosking (1990), Stedinger et al. (1993), Hosking and Wallis (1997)). In comparison to the conventional moments,

L-moments have lower sample variances and are more robust against outliers. For example, L_1 is the same as the population mean, is defined in terms of a conceptual sample of size $r = 1$, while L_2 is an alternative to the population standard deviation, is defined in terms of a conceptual sample of size $r = 2$. Similarly, the L-moments L_3 and L_4 are alternatives to the un-scaled measures of skewness and kurtosis μ_3 and μ_4 , respectively (see Sillitto (1969)). Elamir and Seheult (2003) introduced an extension of L-moments and called them TL-moments (trimmed L-moments). TL-moments are more robust than L-moments and exist even if the distribution does not have a mean, for example the TL-moments exist for Cauchy distribution (see Abdul-Moniem (2009) and Shabri et al. (2011)). Abdul-Moniem (2007) derived L-moments and TL-moments for the exponential distribution. Similar work has been done by Shahzad and Asghar (2013) for Dagum distribution.

2.1 Methodology for L-moments for Lindley Distribution

Let $X_1, X_2, \dots, X_n$ be a random sample from Lindley distribution defined in (1) and $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ denote the corresponding order statistics. Then the r^{th} L-moment defined by Hosking (1990) is as follows:

$$L_r = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r-k:r}), \quad (4)$$

where

$$E(X_{i:r}) = \frac{r!}{(i-1)!(r-i)!} \int_0^\infty x [F(x)]^{i-1} [1-F(x)]^{r-i} f(x) dx. \quad (5)$$

For $r = 1, 2, 3, 4$ in (4), the first four L-moments can be derived as follows:

$$L_1 = E(X_{1:1}) \quad (6)$$

$$L_2 = \frac{1}{2} E(X_{2:2} - X_{1:2}) \quad (7)$$

$$L_3 = \frac{1}{3} E(X_{3:3} - 2X_{2:3} + X_{1:3}) \quad (8)$$

$$L_4 = \frac{1}{4}E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}). \quad (9)$$

The L-moments of the Lindley distribution are obtained by utilizing (6) - (9) as given below:

$$L_1 = \frac{1}{1+\theta} \left(1 + \frac{2}{\theta}\right) \quad (10)$$

$$L_2 = \frac{1}{8\theta(1+\theta)^2} (2\theta^2 + 6\theta + 3) \quad (11)$$

$$L_3 = \frac{1}{108\theta(1+\theta)^3} (18\theta^3 + 72\theta^2 + 57\theta + 19) \quad (12)$$

$$L_4 = \frac{1}{10368\theta(1+\theta)^4} (1152\theta^4 - 5472\theta^3 - 36024\theta^2 - 57812\theta - 29585). \quad (13)$$

In particular, L_1 , L_2 , L_3 and L_4 are population measures of the location, scale, skewness and kurtosis, respectively.

The L-skewness τ_3 and L-kurtosis τ_4 of Lindley distribution will be given by (cf. Shabri et al., 2011)

$$\tau_3 = \frac{L_3}{L_2} = \frac{2(18\theta^3 + 72\theta^2 + 57\theta + 19)}{27(1+\theta)(2\theta^2 + 6\theta + 3)} \quad (14)$$

and

$$\tau_4 = \frac{L_4}{L_2} = \frac{1152\theta^4 - 5472\theta^3 - 36024\theta^2 - 57812\theta - 29585}{1296(1+\theta)^2(2\theta^2 + 6\theta + 3)}. \quad (15)$$

2.2 Sample L-moments and L-Moment estimators

The L-moments can be estimated from the sample order statistics as follows (see Asquith, 2007):

$$l_r = \frac{1}{r \binom{n}{r}} \sum_{i=1}^n \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r-1-k} \binom{n-i}{k} X_{i:n}. \quad (16)$$

From equations (10) and (16) the L-moment estimator for parameter θ will be given as

$$l_1 = \frac{1}{n} \sum_{i=1}^n x_{i:n} = \bar{x} = \frac{1}{\theta(\theta+1)}(\theta+2). \quad (17)$$

By solving, we get

$$\hat{\theta} = \frac{(\bar{x}^2 + 6\bar{x} + 1)^{1/2} - \bar{x} + 1}{2\bar{x}}. \quad (18)$$

2.3 Methodology for TL-moments for Lindley Distribution

Elamir and Seheult (2003) introduced some robust modification of Eq. (4) (and called it as TL-moments) in which $E(X_{r-k:r})$ is replaced by $E(X_{r+t_1-k:r+t_1+t_2})$ for each r where t_1 smallest and t_2 largest are trimmed from the conceptual sample. The following formula gives the r^{th} TL-moments (cf. Elamir and Seheult, 2003):

$$L_r^{(t_1, t_2)} = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r+t_1-k:r+t_1+t_2}). \quad (19)$$

One can observe that TL-moments are more robust than L-moments and exist even if the distribution does not have a mean, for example the TL-moments exist for Cauchy distribution (cf. Abdul-Moniem and Selim, 2009). TL-moments reduce to L-moments when $t_1 = 0$ and $t_2 = 0$.

TL-moments equation (19) with $t_1 = 1$ and $t_2 = 0$ is defined as $L_r^{(1,0)} \equiv L_r^{(1)}$:

$$L_r^{(1)} = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r+1-k:r+1}). \quad (20)$$

For $r = 1, 2, 3, 4$ in (20), the first four L-moments can be derived as follow:

$$L_1^{(1)} = E(X_{2:2}) \quad (21)$$

$$L_2^{(1)} = \frac{1}{2} E(X_{3:3} - X_{2:3}) \quad (22)$$

$$L_3^{(1)} = \frac{1}{3} E(X_{4:4} - 2X_{3:4} + X_{2:4}) \quad (23)$$

$$L_4^{(1)} = \frac{1}{4}E(X_{5:5} - 3X_{4:5} + 3X_{3:5} - X_{2:5}). \quad (24)$$

The TL-moments of the Lindley distribution are obtained by utilizing (21) - (24) and are given below:

$$L_1^{(1)} = \frac{1}{4\theta(1+\theta)^2}(6\theta^2 + 18\theta + 11) \quad (25)$$

$$L_2^{(1)} = \frac{1}{36\theta(1+\theta)^3}(18\theta^3 + 72\theta^2 + 75\theta + 25) \quad (26)$$

$$L_3^{(1)} = \frac{1}{5184\theta(1+\theta)^4}[2016\theta^4 - 4896\theta^3 - 44520\theta^2 - 75140\theta - 38961]. \quad (27)$$

It may be mentioned that one can similarly obtain an expression for $L_4^{(1)}$ which has been omitted here due to its lengthy form.

The TL-skewness $\tau^{(1,0)}$ of the Lindley distribution will be

$$\tau_3^{(1,0)} = \frac{L_3^{(1)}}{L_2^{(1)}} = \frac{2016\theta^4 - 4896\theta^3 - 44520\theta^2 - 75140\theta - 38961}{144(1+\theta)(18\theta^3 + 72\theta^2 + 75\theta + 25)} \quad (28)$$

and one can calculate TL-kurtosis $\tau_4^{(1,0)}$ by solving $\frac{L_4^{(1)}}{L_2^{(1)}}$.

2.4 Sample TL-moments and TL-moment estimators

The TL-moments can be estimated from a sample as linear combination of order statistics. Elamir and Seheult (2003) presented the following estimator for TL-moments:

$$l_r^{(t_1, t_2)} = \frac{1}{r \binom{n}{r+t_1+t_2}} \sum_{i=t_1+1}^{n-t_2} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r+t_1-k-1} \binom{n-i}{k+t_2} x_{i:n}. \quad (29)$$

If we take $t_1 = 1, t_2 = 0$ in Eq. (29), we get

$$l_r^{(1,0)} = \frac{1}{r \binom{n}{r+1}} \sum_{i=2}^n \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r-k} \binom{n-i}{k} x_{i:n}. \quad (30)$$

From Eqs. (25) and (30) for $r = 1$, the TL-moment estimator for parameter θ can be obtained and is given by $l_1^{(1,0)} \equiv l_1$:

$$l_1 = \frac{2}{n(n-1)} \sum_{i=2}^n (i-1)(n-i)x_{i:n} = \frac{6\theta^2 + 18\theta + 11}{4\theta(1+\theta)^2}. \quad (31)$$

By solving, we get

$$\begin{aligned} \hat{\theta} = & \frac{\left(8 l_1^3 + 63 l_1^2 + 3\sqrt{3}(-16 l_1^5 - 189 l_1^4 - 594 l_1^3 - 135 l_1^2)^{1/2} + 135 l_1 + 27\right)^{1/3}}{6 l_1} \\ & + \frac{(16 l_1^2 + 120 l_1 + 36)}{24 l_1 \left(8 l_1^3 + 63 l_1^2 + 3\sqrt{3}(-16 l_1^5 - 189 l_1^4 - 594 l_1^3 - 135 l_1^2)^{1/2} + 135 l_1 + 27\right)^{1/3}} \\ & + \frac{(4 l_1 - 3)}{6 l_1}. \quad (32) \end{aligned}$$

Table 1: Estimated values of parameter θ of Lindley distribution

θ	n=10		n=20		n=50		n=100	
	LMOM	TLMOM	LMOM	TLMOM	LMOM	TLMOM	LMOM	TLMOM
0.2	0.1660	0.0898	0.1921	0.0501	0.1883	0.0197	0.2112	0.0103
	0.1875	0.1145	0.1651	0.1145	0.1913	0.0183	0.1923	0.0096
	0.1949	0.1125	0.1823	0.0447	0.1895	0.0179	0.2133	0.0105
	0.2052	0.1186	0.2007	0.0519	0.2029	0.0197	0.1893	0.0092
1	1.1461	0.6529	0.9934	0.3016	1.0430	0.1241	1.0048	0.0586
	1.1110	0.6582	1.0739	0.3431	1.0343	0.1191	1.0472	0.0613
	1.0154	0.6327	1.0237	0.3145	1.0264	0.1255	0.9659	0.0548
	1.0142	0.5997	0.9949	0.2913	1.0198	0.1267	0.9851	0.0602
4	4.2748	2.7727	4.1370	1.3879	4.1490	0.5399	3.8104	0.2644
	3.9161	2.5845	3.9207	1.2632	4.2069	0.5884	4.0022	0.2935
	3.8439	2.4434	4.2112	1.3795	4.2940	0.5828	4.1097	0.2954
	4.1496	3.0283	4.1409	1.3161	3.9553	0.5560	3.9105	0.2843
6	5.9445	3.6856	6.4453	2.0486	6.0043	0.8466	5.9278	0.4229
	6.3973	4.0979	6.0913	2.1146	6.2040	0.8935	5.8505	0.4265
	6.2840	3.8693	6.0407	1.9103	5.8188	0.8091	6.2411	0.4756
	5.7637	3.9766	6.4554	2.1008	6.4949	0.9088	6.0251	0.4286

2.5 Conclusion and Discussion

We have derived L-moments and TL-moments for Lindley distribution and compared parameter estimates obtained by both methods. For parameter estimation, we found

that L-moments provide better fit and are more efficient than TL-moments for Lindley distribution because TL-moments loose some information of data when there are no outliers in the data. As we increase the sample size we get better estimates with less bias for estimator $\hat{\theta}$. In order to verify these results, we have drawn samples of size $n = 10, 20, 50$ and 100 with 10000 replications. With the help of MATLAB programming, parameter estimates obtained by L-moments(LMOM) and TL-moments (TLMOM) of unknown parameter θ by using equations (18) and (32) are computed for Lindley distribution. Table 1 presents best four values of the estimates of θ based on LMOM and TLMOM methods using different values of θ such as $0.2, 1, 4$ and 6 .

3 Doubly Truncated Lindley Distribution

The pdf of the doubly truncated Lindley distribution is given by

$$f(x) = \left(\frac{1}{P-Q} \right) \frac{\theta^2}{1+\theta} (1+x)e^{-\theta x}, \quad Q_1 \leq x \leq P_1, \quad \theta > 0, \quad (33)$$

and the cdf is given by

$$F(x) = \beta - \left(\frac{1}{P-Q} \right) \frac{e^{-\theta x}}{1+\theta} [\theta(1+x) + 1], \quad Q_1 \leq x \leq P_1, \quad (34)$$

where

$$\beta = \frac{1}{(P-Q)} \frac{e^{-\theta Q_1}}{(1+\theta)} [\theta(1+Q_1) + 1], \quad (35)$$

$$Q = \frac{1}{(1+\theta)} \left[(\theta-1) + e^{-\theta Q_1} \{1 - \theta(1+Q_1)\} \right], \quad (36)$$

$$P = \frac{1}{(1+\theta)} \left[(\theta-1) + e^{-\theta P_1} \{1 - \theta(1+P_1)\} \right], \quad (37)$$

and Q and $(1-P)$, ($0 < Q < P < 1$) are the proportions of truncation on the left and right of the Lindley distribution.

It may be noted that by letting $Q \rightarrow 0$ and $P \rightarrow 1$ (or, equivalently, $Q_1 \rightarrow 0$ and $P_1 \rightarrow \infty$), the distribution in (33) reduces to that of the untruncated Lindley distribution given in (1).

The characterizing differential equation for doubly truncated Lindley distribution is given by

$$(1 + \theta + \theta x)f(x) = \alpha(1 + x) + \theta^2[1 - F(x)] + \theta^2 x[1 - F(x)], \quad (38)$$

where $\alpha = \theta^2(\beta - 1)$ and β is as given in (35).

3.1 Recurrence Relations for Higher Moments of Order Statistics from Doubly Truncated Lindley Distribution

The joint density function of $X_{r_1:n}, X_{r_2:n}, \dots$, and $X_{r_k:n} (1 \leq r_1 < r_2 < \dots < r_k \leq n)$ for a doubly truncated distribution is given by

$$\begin{aligned} f_{r_1, r_2, \dots, r_k:n}(x_1, \dots, x_k) &= C_{r_1, r_2, \dots, r_k:n} [F(x_1)]^{r_1-1} [F(x_2) - F(x_1)]^{r_2-r_1-1} \dots \\ &\quad [F(x_k) - F(x_{k-1})]^{r_k-r_{k-1}-1} [1 - F(x_k)]^{n-r_k} \\ &\quad f(x_1)f(x_2) \dots f(x_k), Q_1 \leq x_1 < \dots < x_k \leq P_1, \end{aligned} \quad (39)$$

where

$$C_{r_1, r_2, \dots, r_k:n} = \frac{n!}{(r_1 - 1)!(r_2 - r_1 - 1)! \dots (r_k - r_{k-1} - 1)!(n - r_k)!}$$

[cf. David and Nagaraja, 2003, p.12], and $f(x)$ and $F(x)$ are as given in equations (33) and (34), respectively. Then by making use of the characterizing differential equation in (38), we establish recurrence relations for the product moments of k order statistics arising from doubly truncated Lindley distribution.

Theorem 3.1. For $1 \leq r_1 < r_2 < \dots < r_k \leq n$, $r_k \geq r_{k-1} + 2$,

$$\begin{aligned} (1 + \theta)\mu_{r_1, r_2, \dots, r_k:n}^{(i_1, i_2, \dots, i_k)} &= \alpha n \left[\frac{1}{i_k + 1} \left\{ \mu_{r_1, r_2, \dots, r_k:n-1}^{(i_1, i_2, \dots, i_k+1)} - \mu_{r_1, r_2, \dots, r_{k-1}:n-1}^{(i_1, i_2, \dots, i_k+1)} \right\} \right. \\ &\quad \left. + \frac{1}{i_k + 2} \left\{ \mu_{r_1, r_2, \dots, r_k:n-1}^{(i_1, i_2, \dots, i_k+2)} - \mu_{r_1, r_2, \dots, r_{k-1}:n-1}^{(i_1, i_2, \dots, i_k+2)} \right\} \right] \\ &\quad + \theta^2 (n - r_k + 1) \left[\frac{1}{i_k + 1} \left\{ \mu_{r_1, r_2, \dots, r_k:n}^{(i_1, i_2, \dots, i_k+1)} - \mu_{r_1, r_2, \dots, r_{k-1}:n}^{(i_1, i_2, \dots, i_k+1)} \right\} \right. \\ &\quad \left. + \frac{1}{i_k + 2} \left\{ \mu_{r_1, r_2, \dots, r_k:n}^{(i_1, i_2, \dots, i_k+2)} - \mu_{r_1, r_2, \dots, r_{k-1}:n}^{(i_1, i_2, \dots, i_k+2)} \right\} \right] \\ &\quad - \theta \mu_{r_1, r_2, \dots, r_k:n}^{(i_1, i_2, \dots, i_k+1)}. \end{aligned} \quad (40)$$

where $\alpha = \theta^2(\beta - 1)$ as defined in (38).

Proof. From equation (39), we have for $1 \leq r_1 < r_2 < \dots < r_k \leq n$,

$$\begin{aligned}
& (1 + \theta)\mu_{r_1, r_2, \dots, r_k: n}^{(i_1, i_2, \dots, i_k)} + \theta\mu_{r_1, r_2, \dots, r_k: n}^{(i_1, i_2, \dots, i_k+1)} = \\
& C_{r_1, r_2, \dots, r_k: n} \int_{Q_1}^{P_1} \int_{x_1}^{P_1} \dots \int_{x_{k-1}}^{P_1} x_1^{i_1} x_2^{i_2} \dots x_k^{i_k} [F(x_1)]^{r_1-1} \\
& [F(x_2) - F(x_1)]^{r_2-r_1-1} \dots [1 - F(x_k)]^{n-r_k} \\
& (1 + \theta + \theta x_k) f(x_1) f(x_2) \dots f(x_k) dx_k \dots dx_1 \\
& = C_{r_1, r_2, \dots, r_k: n} \int_{Q_1}^{P_1} \int_{x_1}^{P_1} \dots \int_{x_{k-2}}^{P_1} x_1^{i_1} x_2^{i_2} \dots x_{k-1}^{i_{k-1}} [F(x_1)]^{r_1-1} [F(x_2) - F(x_1)]^{r_2-r_1-1} \dots \\
& [F(x_{k-1}) - F(x_{k-2})]^{r_{k-1}-r_{k-2}-1} I(x_k) f(x_1) \dots f(x_{k-1}) dx_{k-1} \dots dx_2 dx_1, \quad (41)
\end{aligned}$$

where

$$I(x_k) = \int_{x_{k-1}}^{P_1} x_k^{i_k} [F(x_k) - F(x_{k-1})]^{r_k-r_{k-1}-1} [1 - F(x_k)]^{n-r_k} (1 + \theta + \theta x_k) f(x_k) dx_k.$$

Making use of characterizing differential equation (38), we have

$$\begin{aligned}
I(x_k) = & \alpha \left[\int_{x_{k-1}}^{P_1} x_k^{i_k} [F(x_k) - F(x_{k-1})]^{r_k-r_{k-1}-1} [1 - F(x_k)]^{n-r_k} dx_k \right. \\
& + \int_{x_{k-1}}^{P_1} x_k^{i_k+1} [F(x_k) - F(x_{k-1})]^{r_k-r_{k-1}-1} [1 - F(x_k)]^{n-r_k} dx_k \Big] \\
& + \theta^2 \left[\int_{x_{k-1}}^{P_1} x_k^{i_k} [F(x_k) - F(x_{k-1})]^{r_k-r_{k-1}-1} [1 - F(x_k)]^{n-r_k+1} dx_k \right. \\
& + \int_{x_{k-1}}^{P_1} x_k^{i_k+1} [F(x_k) - F(x_{k-1})]^{r_k-r_{k-1}-1} [1 - F(x_k)]^{n-r_k+1} dx_k \Big]
\end{aligned}$$

$$= \alpha [E(x_k; 0, 0) + E(x_k; 1, 0)] + \theta^2 [E(x_k; 0, 1) + E(x_k; 1, 1)], \quad (42)$$

where

$$E(x_k; a, b) = \int_{x_{k-1}}^{P_1} x_k^{i_k+a} [F(x_k) - F(x_{k-1})]^{r_k-r_{k-1}-1} [1 - F(x_k)]^{n-r_k+b} dx_k,$$

$$a = 0, 1 \text{ and } b = 0, 1.$$

Integrating by parts, treating $x_k^{i_k+a}$ for integration and rest of the integrand for differentiation and then substituting for $E(x_k; 0, 0)$, $E(x_k; 1, 0)$, $E(x_k; 0, 1)$ and $E(x_k; 1, 1)$ in (42) and then substituting the resulting expression for $I(x_k)$ in equation (3.9) and simplifying, we derive the relation in (40).

□

Proceeding on similar lines, one can derive the following recurrence relations.

Theorem 3.2. For $1 \leq r_1 < r_2 < \dots < r_k \leq n$, $r_k = r_{k-1} + 1$,

$$\begin{aligned} (1 + \theta) \mu_{r_1, r_2, \dots, r_{k-1}, r_{k-1}+1; n}^{(i_1, i_2, \dots, i_k)} = \\ \alpha n \left[\frac{1}{i_k + 1} \left\{ \mu_{r_1, r_2, \dots, r_{k-1}, r_{k-1}+1; n-1}^{(i_1, i_2, \dots, i_k+1)} - \mu_{r_1, r_2, \dots, r_{k-1}; n-1}^{(i_1, i_2, \dots, i_{k-1}+i_k+1)} \right\} \right. \\ \left. + \frac{1}{i_k + 2} \left\{ \mu_{r_1, r_2, \dots, r_{k-1}, r_{k-1}+1; n-1}^{(i_1, i_2, \dots, i_k+2)} - \mu_{r_1, r_2, \dots, r_{k-1}; n-1}^{(i_1, i_2, \dots, i_{k-1}+i_k+2)} \right\} \right] \\ + \theta^2 (n - r_{k-1}) \left[\frac{1}{i_k + 1} \left\{ \mu_{r_1, r_2, \dots, r_{k-1}, r_{k-1}+1; n}^{(i_1, i_2, \dots, i_k+1)} - \mu_{r_1, r_2, \dots, r_{k-1}; n}^{(i_1, i_2, \dots, i_{k-1}+i_k+1)} \right\} \right. \\ \left. + \frac{1}{i_k + 2} \left\{ \mu_{r_1, r_2, \dots, r_{k-1}, r_{k-1}+1; n}^{(i_1, i_2, \dots, i_k+2)} - \mu_{r_1, r_2, \dots, r_{k-1}; n}^{(i_1, i_2, \dots, i_{k-1}+i_k+2)} \right\} \right] \\ - \theta \mu_{r_1, r_2, \dots, r_{k-1}, r_{k-1}+1; n}^{(i_1, i_2, \dots, i_k+1)}. \end{aligned} \quad (43)$$

Theorem 3.3. For $r_k = n$,

$$\begin{aligned} (1 + \theta) \mu_{r_1, r_2, \dots, r_{k-1}, n; n}^{(i_1, i_2, \dots, i_k)} = \\ \alpha n \left[\frac{1}{i_k + 1} \left\{ P_1^{i_k+1} \mu_{r_1, r_2, \dots, r_{k-1}; n-1}^{(i_1, i_2, \dots, i_{k-1})} - \mu_{r_1, r_2, \dots, r_{k-1}, n-1; n-1}^{(i_1, i_2, \dots, i_k+1)} \right\} \right. \\ \left. - \frac{1}{i_k + 2} \left\{ P_1^{i_k+1} \mu_{r_1, r_2, \dots, r_{k-1}; n-1}^{(i_1, i_2, \dots, i_{k-1})} - \mu_{r_1, r_2, \dots, r_{k-1}, n-1; n-1}^{(i_1, i_2, \dots, i_k+2)} \right\} \right] \\ + \theta^2 \left[\frac{1}{i_k + 1} \left\{ \mu_{r_1, r_2, \dots, r_{k-1}, n; n}^{(i_1, i_2, \dots, i_k+1)} - \mu_{r_1, r_2, \dots, r_{k-1}, n-1; n}^{(i_1, i_2, \dots, i_k+1)} \right\} \right. \\ \left. + \frac{1}{i_k + 2} \left\{ \mu_{r_1, r_2, \dots, r_{k-1}, n; n}^{(i_1, i_2, \dots, i_k+2)} - \mu_{r_1, r_2, \dots, r_{k-1}, n-1; n}^{(i_1, i_2, \dots, i_k+2)} \right\} \right] - \theta \mu_{r_1, r_2, \dots, r_{k-1}, n; n}^{(i_1, i_2, \dots, i_k+1)}. \end{aligned} \quad (44)$$

3.2 Relations for untruncated Lindley Distribution

By taking $Q \rightarrow 0$ and $(1 - P) \rightarrow 0 (\Rightarrow Q_1 \rightarrow 0, P_1 \rightarrow \infty)$ in Theorems 3.1-3.3, one can deduce the corresponding results for higher moments of order statistics from untruncated Lindley distribution (1), which are quoted below:

Theorem 3.4. For $r_2 - r_1 \geq 2$ and $i_1, i_2, \dots, i_k > 0$,

$$(1 + \theta)\mu_{r_1, r_2, \dots, r_k; n}^{(i_1, i_2, \dots, i_k)} = \theta^2(n - r_k + 1) \left[\frac{1}{i_k + 1} \left\{ \mu_{r_1, r_2, \dots, r_k; n}^{(i_1, i_2, \dots, i_k + 1)} - \mu_{r_1, r_2, \dots, r_k - 1; n}^{(i_1, i_2, \dots, i_k + 1)} \right\} \right. \\ \left. + \frac{1}{i_k + 2} \left\{ \mu_{r_1, r_2, \dots, r_k; n}^{(i_1, i_2, \dots, i_k + 2)} - \mu_{r_1, r_2, \dots, r_k - 1; n}^{(i_1, i_2, \dots, i_k + 2)} \right\} \right] - \theta \mu_{r_1, r_2, \dots, r_k; n}^{(i_1, i_2, \dots, i_k + 1)}. \quad (45)$$

Theorem 3.5. For $1 \leq r_1 < r_2 < \dots < r_k \leq n$, $r_k = r_{k-1} + 1$,

$$(1 + \theta)\mu_{r_1, r_2, \dots, r_{k-1}, r_{k-1} + 1; n}^{(i_1, i_2, \dots, i_k)} = \\ \theta^2(n - r_{k-1}) \left[\frac{1}{i_k + 1} \left\{ \mu_{r_1, r_2, \dots, r_{k-1}, r_{k-1} + 1; n}^{(i_1, i_2, \dots, i_k + 1)} - \mu_{r_1, r_2, \dots, r_{k-2}, r_{k-1}; n}^{(i_1, i_2, \dots, i_{k-1} + i_k + 1)} \right\} \right. \\ \left. + \frac{1}{i_k + 2} \left\{ \mu_{r_1, r_2, \dots, r_{k-1}, r_{k-1} + 1; n}^{(i_1, i_2, \dots, i_k + 2)} - \mu_{r_1, r_2, \dots, r_{k-1}; n}^{(i_1, i_2, \dots, i_{k-1} + i_k + 2)} \right\} \right] \\ - \theta \mu_{r_1, r_2, \dots, r_{k-1}, r_{k-1} + 1; n}^{(i_1, i_2, \dots, i_k + 1)}. \quad (46)$$

Theorem 3.6. For $1 \leq r_1 < r_2 < \dots < r_k \leq n$, $r_k = n$,

$$(1 + \theta)\mu_{r_1, r_2, \dots, r_{k-1}, n; n}^{(i_1, i_2, \dots, i_k)} = \theta^2 \left[\frac{1}{i_k + 1} \left\{ \mu_{r_1, r_2, \dots, r_{k-1}, n; n}^{(i_1, i_2, \dots, i_k + 1)} - \mu_{r_1, r_2, \dots, r_{k-1}, n-1; n}^{(i_1, i_2, \dots, i_k + 1)} \right\} \right. \\ \left. + \frac{1}{i_k + 2} \left\{ \mu_{r_1, r_2, \dots, r_{k-1}, n; n}^{(i_1, i_2, \dots, i_k + 2)} - \mu_{r_1, r_2, \dots, r_{k-1}, n-1; n}^{(i_1, i_2, \dots, i_k + 2)} \right\} \right] - \theta \mu_{r_1, r_2, \dots, r_{k-1}, n; n}^{(i_1, i_2, \dots, i_k + 1)}. \quad (47)$$

Remark 3.1. From the results of higher order moments of order k , derived above in Section 3.2, for Lindley distribution defined in (1), one can easily deduce the results for lower order moments of order $m (< k)$ by setting $i_1 = i_2 = \dots = i_{k-m} = 0$. Hence, the recurrence relations for single, double (product), triple and quadruple moments of order statistics arising from Lindley distribution (1) can be obtained by taking $m = 1, 2, 3, 4$, respectively. It may be mentioned that the results so deduced for single and product moments (i.e., for the case $m = 1$ and 2) of order statistics from Lindley distribution are in agreement with the corresponding results obtained by Athar et al. (2014, Remark 2.1, pp.4-5).

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