

CONDITIONAL DISTRIBUTIONS AND RELIABILITY ANALYSIS ASSOCIATED WITH MARSHALL-OLKIN BIVARIATE EXPONENTIAL DISTRIBUTION

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ABSTRACT

By considering the bivariate exponential distribution due to Marshall and Olkin (1967), we derive the conditional distributions and regression equations. It is observed that the regression equations are non-linear. An alternative derivation for the moment generating function using three independent exponential distributions is also given. Assuming that the joint distribution of the component failure times is bivariate exponential, we obtain the reliability measures for two-unit standby, parallel and series systems.

Key words and Phrases: *Conditional distribution, Moment generating function, Regression, Reliability measures, Weighted average.*

1 Introduction

There is an extensive literature on the construction of bivariate exponential models. Several authors proposed different bivariate exponential models.

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Marshall and Olkin (1967) proposed a multivariate exponential distribution having dependent exponential marginals which characterizes a fatal shock model. The distribution does not possess the probability density function (pdf) with respect to the Lebesgue measure in R^2 . Bemis *et al.* (1972) proposed the pdf with respect to an appropriate measure and derived some properties. Paul Rajamanickam and Chandrasekar (1998) derived the asymptotic confidence interval for the steady-state availability of a one unit system with dependent structure having Marshall-Olkin bivariate exponential (MOBVE) joint distribution for the failure and repair times. Exact distributional properties of MOBVE are rare in the literature.

This paper studies the properties of MOBVE distribution through the pdf approach. Section 2 introduces the distribution and the pdf due to Bemis *et al.* (1972). The conditional distributions and the regression equations are also obtained and it is observed that the regression equations are not linear. Section 3 provides an alternative derivation for the moment generating function (mgf) using three independent exponential distributions. Section 4 derives the performance measures of two-unit standby, parallel and series systems assuming that the joint distribution of the component failure times is MOBVE. In the case of standby and parallel systems, it is observed that the system failure times are the weighted averages of exponential distributions.

2 Conditional distributions and Regression

In this section we observe that the regression equations are not linear.

The survival function of the MOBVE distribution is given by

$$\bar{F}(x, y) = e^{-\lambda_1 x - \lambda_2 y - \lambda_{12}(x \vee y)}; \quad x, y > 0; \lambda_1, \lambda_2 > 0, \lambda_{12} \geq 0.$$

Here $a \vee b = \max\{a, b\}$.

Bemis *et al.* (1972) proposed a pdf as follows:

$$f(x, y) = \begin{cases} \lambda_1(\lambda_2 + \lambda_{12})e^{-\lambda_1 x - (\lambda_2 + \lambda_{12})y}, & x < y \\ \lambda_2(\lambda_1 + \lambda_{12})e^{-(\lambda_1 + \lambda_{12})x - \lambda_2 y}, & x > y \\ \lambda_{12}e^{-(\lambda_1 + \lambda_2 + \lambda_{12})x}, & x = y. \end{cases} \quad (2.1)$$

Remark 2.1. The pdf can be viewed as a mixture of three pdfs with the mixing proportions $\frac{\lambda_1}{\lambda}$, $\frac{\lambda_2}{\lambda}$ and $\frac{\lambda_{12}}{\lambda}$, where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. In fact,

$$f(x, y) = \frac{\lambda_1}{\lambda} f_1(x, y) + \frac{\lambda_2}{\lambda} f_2(x, y) + \frac{\lambda_{12}}{\lambda} f_3(x, y),$$

where

$$f_1(x, y) = \begin{cases} \lambda(\lambda_2 + \lambda_{12})e^{-\lambda_1 x - (\lambda_2 + \lambda_{12})y}, & 0 < x < y < \infty \\ 0, & \text{otherwise} \end{cases},$$

$$f_2(x, y) = \begin{cases} \lambda(\lambda_1 + \lambda_{12})e^{-(\lambda_1 + \lambda_{12})x - \lambda_2 y}, & 0 < y < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_3(x, y) = \begin{cases} \lambda e^{-\lambda x}, & 0 < x = y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.1. The conditional pdf of X given $Y = y$ is

$$f(x | y) = \lambda_1 e^{-\lambda_1 x} I_{(0, y)}(x) + \frac{\lambda_{12}}{(\lambda_2 + \lambda_{12})} e^{-\lambda_1 x} I_{\{y\}}(x) \\ + \frac{\lambda_2(\lambda_1 + \lambda_{12})}{(\lambda_2 + \lambda_{12})} e^{-(\lambda_1 + \lambda_{12})x + \lambda_{12}y} I_{(y, \infty)}(x), x > 0.$$

Proof. The joint pdf of (X, Y) is given in (2.1) and the pdf of Y is

$$f_2(y) = (\lambda_2 + \lambda_{12})e^{-(\lambda_2 + \lambda_{12})y}, y > 0.$$

The proof follows from the fact that

$$f(x | y) = \frac{f(x, y)}{f_2(y)}, x > 0.$$

Corollary 2.1.

$$(i) E(X | Y = y) = \frac{1}{\lambda_1} - \frac{\lambda \lambda_{12}}{\lambda_1(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})} e^{-\lambda_1 y}$$

and

$$(ii) E(Y | X = x) = \frac{1}{\lambda_2} - \frac{\lambda \lambda_{12}}{\lambda_2(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})} e^{-\lambda_2 x}.$$

Proof: From Theorem 2.1,

$$\begin{aligned}
 E(X | Y = y) &= \int_0^\infty x f(x | y) dx \\
 &= \int_0^y x \lambda_1 e^{-\lambda_1 x} dx \\
 &\quad + \int_y^\infty x \frac{\lambda_2 (\lambda_1 + \lambda_{12})}{(\lambda_2 + \lambda_{12})} e^{-(\lambda_1 + \lambda_{12})x + \lambda_{12}y} dx \\
 &\quad + y \frac{\lambda_{12}}{(\lambda_2 + \lambda_{12})} e^{-\lambda_1 y} \\
 &= -y e^{-\lambda_1 y} + \frac{1}{\lambda_1} (1 - e^{-\lambda_1 y}) \\
 &\quad + \frac{\lambda_2}{(\lambda_2 + \lambda_{12})} e^{\lambda_{12}y} \left\{ y e^{-(\lambda_1 + \lambda_{12})y} + \frac{1}{(\lambda_1 + \lambda_{12})} e^{-(\lambda_1 + \lambda_{12})y} \right\} \\
 &\quad + y \frac{\lambda_{12}}{(\lambda_2 + \lambda_{12})} e^{-\lambda_1 y} \\
 &= \frac{1}{\lambda_1} - \frac{\lambda \lambda_{12}}{\lambda_1 (\lambda_1 + \lambda_{12}) (\lambda_2 + \lambda_{12})} e^{-\lambda_1 y}.
 \end{aligned}$$

This proves (i). Similar arguments lead to (ii). **Remark 2.2.** The regression equations are

$$X = \frac{1}{\lambda_1} - \frac{\lambda \lambda_{12}}{\lambda_1 (\lambda_1 + \lambda_{12}) (\lambda_2 + \lambda_{12})} e^{-\lambda_1 Y}$$

and

$$Y = \frac{1}{\lambda_2} - \frac{\lambda \lambda_{12}}{\lambda_2 (\lambda_1 + \lambda_{12}) (\lambda_2 + \lambda_{12})} e^{-\lambda_2 X}.$$

The regression equations are not linear.

Remark 2.3. (i) The mgf of the conditional distribution of X given $Y = y$ is

$$M(t | y) = \frac{\lambda_1}{(\lambda_1 - t)} + \frac{\lambda_{12} t (t - \lambda) e^{-(\lambda_1 - t)y}}{(\lambda_2 + \lambda_{12})(\lambda_1 - t)(\lambda_1 + \lambda_{12} - t)}, t < \lambda_1.$$

(ii) The mgf of the conditional distribution of Y given $X = x$ is

$$M(t | x) = \frac{\lambda_2}{(\lambda_2 - t)} + \frac{\lambda_{12} t (t - \lambda) e^{-(\lambda_2 - t)x}}{(\lambda_1 + \lambda_{12})(\lambda_2 - t)(\lambda_2 + \lambda_{12} - t)}, t < \lambda_2.$$

3 An alternative derivation for the moment generating function

Marshall and Olkin (1967) derived the mgf using pdf approach. Here we derive the mgf using three independent exponential distributions.

Theorem 3.1. The mgf of (X, Y) is

$$M(t_1, t_2) = \frac{(\lambda - t_1 - t_2)(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12}) + \lambda_{12}t_1t_2}{(\lambda - t_1 - t_2)(\lambda_1 + \lambda_{12} - t_1)(\lambda_2 + \lambda_{12} - t_2)}, \quad (3.1)$$

$$t_1 < \lambda_1 + \lambda_{12}, t_2 < \lambda_2 + \lambda_{12}, t_1 + t_2 < \lambda.$$

Proof. Let U_1, U_2 and U_3 be three independent exponential random variables with the parameters λ_1, λ_2 and λ_{12} respectively. Define $X = U_1 \wedge U_3$ and $Y = U_2 \wedge U_3$, where $a \wedge b = \min\{a, b\}$. Then (X, Y) follows MOBE distribution (See Barlow and Proschan(1975)).

Thus

$$\begin{aligned} M(t_1, t_2) &= E \left\{ e^{t_1(U_1 \wedge U_3) + t_2(U_2 \wedge U_3)} \right\} \\ &= \int_0^\infty \int_0^\infty \int_0^\infty e^{t_1(u_1 \wedge u_3) + t_2(u_2 \wedge u_3)} \lambda_1 \lambda_2 \lambda_{12} e^{-\lambda_1 u_1 - \lambda_2 u_2 - \lambda_{12} u_3} du_1 du_2 du_3 \\ &= \int_0^\infty \int_{u_1}^\infty \int_{u_2}^\infty e^{t_1 u_1 + t_2 u_2} \lambda_1 \lambda_2 \lambda_{12} e^{-\lambda_1 u_1 - \lambda_2 u_2 - \lambda_{12} u_3} du_3 du_2 du_1 \\ &\quad + \int_0^\infty \int_{u_2}^\infty \int_{u_1}^\infty e^{t_1 u_1 + t_2 u_2} \lambda_1 \lambda_2 \lambda_{12} e^{-\lambda_1 u_1 - \lambda_2 u_2 - \lambda_{12} u_3} du_3 du_1 du_2 \\ &\quad + \int_0^\infty \int_{u_1}^\infty \int_{u_3}^\infty e^{t_1 u_1 + t_2 u_3} \lambda_1 \lambda_2 \lambda_{12} e^{-\lambda_1 u_1 - \lambda_2 u_2 - \lambda_{12} u_3} du_2 du_3 du_1 \\ &\quad + \int_0^\infty \int_{u_2}^\infty \int_{u_3}^\infty e^{t_1 u_3 + t_2 u_2} \lambda_1 \lambda_2 \lambda_{12} e^{-\lambda_1 u_1 - \lambda_2 u_2 - \lambda_{12} u_3} du_1 du_3 du_2 \\ &\quad + \int_0^\infty \int_{u_3}^\infty \int_{u_1}^\infty e^{t_1 u_3 + t_2 u_3} \lambda_1 \lambda_2 \lambda_{12} e^{-\lambda_1 u_1 - \lambda_2 u_2 - \lambda_{12} u_3} du_2 du_1 du_3 \\ &\quad + \int_0^\infty \int_{u_3}^\infty \int_{u_2}^\infty e^{t_1 u_3 + t_2 u_3} \lambda_1 \lambda_2 \lambda_{12} e^{-\lambda_1 u_1 - \lambda_2 u_2 - \lambda_{12} u_3} du_1 du_2 du_3 \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \text{ say.} \end{aligned}$$

Then

$$\begin{aligned}
 I_1 &= \frac{\lambda_1 \lambda_2}{(\lambda_2 + \lambda_{12} - t_2)(\lambda - t_1 - t_2)}, \\
 I_2 &= \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_{12} - t_1)(\lambda - t_1 - t_2)}, \\
 I_3 &= \frac{\lambda_1 \lambda_{12}}{(\lambda_2 + \lambda_{12} - t_2)(\lambda - t_1 - t_2)}, \\
 I_4 &= \frac{\lambda_2 \lambda_{12}}{(\lambda_1 + \lambda_{12} - t_1)(\lambda - t_1 - t_2)}, \\
 I_5 &= \frac{\lambda_1 \lambda_{12}}{(\lambda_2 + \lambda_2)(\lambda - t_1 - t_2)}, \text{ and} \\
 I_6 &= \frac{\lambda_2 \lambda_{12}}{(\lambda_1 + \lambda_{12})(\lambda - t_1 - t_2)}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 M(t_1, t_2) &= (I_1 + I_3) + (I_2 + I_4) + (I_5 + I_6) \\
 &= \frac{\lambda_1(\lambda_2 + \lambda_{12})}{(\lambda_2 + \lambda_{12} - t_2)(\lambda - t_1 - t_2)} + \frac{\lambda_2(\lambda_1 + \lambda_{12})}{(\lambda_1 + \lambda_{12} - t_1)(\lambda - t_1 - t_2)} + \frac{\lambda_{12}}{(\lambda - t_1 - t_2)} \\
 &= \frac{(\lambda - t_1 - t_2)(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12}) + \lambda_{12}t_1t_2}{(\lambda - t_1 - t_2)(\lambda_1 + \lambda_{12} - t_1)(\lambda_2 + \lambda_{12} - t_2)},
 \end{aligned}$$

$$t_1 < \lambda_1 + \lambda_{12}, t_2 < \lambda_2 + \lambda_{12}, t_1 + t_2 < \lambda.$$

Remark 3.1. The above expression coincides with the one given in Barlow and Proschan (1975).

4 Reliability measures

Consider a two-unit system with the failure time distributions following a MOBE. By considering standby, parallel and series systems, we derive the system failure times and the performance measures of these systems.

4.1 Standby system

If $\lambda_{12} = 0$, the failure times are independent and the results are known in the literature, (see Rau (1970)). So we assume that $\lambda_{12} > 0$ throughout the section. Four cases arise according as (i) $\lambda_2 - \lambda_1 - \lambda_{12} \neq 0$, $\lambda_1 - \lambda_2 - \lambda_{12} \neq 0$ (ii) $\lambda_2 - \lambda_1 - \lambda_{12} \neq 0$,