Journal of the Kerala Statistical Association
Vol.34, December 2023, 52-83 *ISSN 2249-4553*

A Versatile Probabilistic Model based on Yun-G Family of Distributions and its Applications in Engineering Sector

Akhila P.[1], Girish Babu M.*[1], and Hassan S. Bakouch [2,3]

[1]Department of Statistics, Govt. Arts and Science College, Kozhikode, India.
[2]Department of Mathematics, College of Science, Qassim University, Buraydah, Saudi Arabia.
[3]Department of Mathematics, Faculty of Science, Tanta University, Tanta 31111, Egypt.

ABSTRACT

The present paper discusses a versatile three -parameter distribution from the Yun-G family. The important statistical properties like moments, stochastic ordering, and entropy are studied in this paper. Two characterizations of the distribution are obtained using the hazard rate function and truncated moments. The statistical inference of the distribution is studied by executing five different methods of parameter estimation, such as maximum likelihood estimation, ordinary least square method, weighted least square method, Cramér-von Mises method, and Anderson–Darling method. To study the fitting and applicability of the proposed distribution, two real life data sets from the engineering sector were analyzed and the proposed distribution is found to be more appropriate than the other competitive distributions. A comprehensive simulation study is also conducted and it showed the accuracy and consistency of the estimation techniques.

Key words and Phrases: *Estimation; Fréchet distribution; Hazard rate function; Simulation; Statistical model; Yun transform.*

*giristat@gmail.com

1 Introduction

In this data-driven era, the velocity of data and the complexity of its organization make flexible distributions essential. The invention of normal distribution got wide recognition due to its simple properties and central limit theorem. However, researchers face a challenge when it comes to practical applications of skewed or extremely valued data. Discoveries of the new family of distributions have shown how it can be used to model data in applied science, especially in engineering. The operational and reliability data sets are crucial for developing predictive models, validating simulations, training machine learning algorithms, and making data-driven decisions in engineering projects. Many families of distributions have been widely used for data modeling across various domains during the last few decades, see Marshall and Olkin (1997), Alzaatreh et al. (2013), and among others.

The Fréchet distribution or the inverse Weibull distribution was introduced by French Mathematician Maurice Fréchet in 1927. This distribution is a special case of the generalized extreme value distribution. Thus, it is also known as extreme value distribution Type II. This distribution is commonly used to model a wide range of real phenomena with heavy tails like sea wave dynamics, earthquakes, engineering, actuarial science, and other events. The distribution function and density function of Fréchet distribution are, respectively,

$$G_{\sigma,\lambda}(x) = e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}, x > 0, \tag{1.1}$$

and

$$g_{\sigma,\lambda}(x) = \lambda\sigma^{\lambda}x^{-(\lambda+1)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}, x > 0, \tag{1.2}$$

where $\sigma > 0$ is the scale parameter and $\lambda > 0$ is the shape parameter.

From the literature, it is clear that conventional probability distributions like Fréchet are not universally suitable for modeling all types of real world phenomena. Hence, it is crucial to suggest novel distributions that offer enhanced flexibility in representing real world data sets. This can be achieved by expanding the traditional distributions through the introduction of interpretable parameters or by creating new families of distribution.

The realm of statistical literature encompasses numerous extensions of the Fréchet distributions. Recent generalizations of Fréchet distributions are; Generalization of Fréchet distribution by Jayakumar and Babu (2019), cubic transmuted Fréchet distribution by

Shalabi (2020), the generalized odd log-logistic Fréchet distribution by Abd El Khalek and Raina (2022), the inverted Gompertz-Fréchet distribution proposed by Akarawak et al. (2023), Marshall-Olkin exponentiated Fréchet distribution by Aurise et al. (2023) and the novel Kumaraswamy power Fréchet distribution by Najwan et al. (2023), and among others.

The current study is inspired by a newly proposed Yun-G family of distribution by Chesneau et al. (2021) leveraging a mathematical transform introduced by Yun (2014). The unique features of this transform such as continuous derivatives, and simple expression of the inverse function serve as key motivators for this study.

Yun transform is defined as;

$$T_\alpha(x) = \frac{(1+x)^\alpha - (1-x)^\alpha}{(1+x)^\alpha + (1-x)^\alpha}, \qquad x \in [0,1], \alpha \geq 1, \tag{1.3}$$

with $T_\alpha(x) = 0$ for $x < 0$ and $T_\alpha(x) = 1$ for $x > 1$. Then, $T_\alpha(x)$ has the properties of a distribution function.
The probability density function (pdf) is given by;

$$t_\alpha(x) = 4\alpha \frac{(1-x^2)^{\alpha-1}}{((1+x)^\alpha + (1-x)^\alpha)^2}, \qquad x \in [0,1]. \tag{1.4}$$

The hazard rate function (hrf) is written as

$$h_\alpha(x) = \frac{t_\alpha(x)}{1 - T_\alpha(x)} = 2\alpha \frac{(1+x)^\alpha}{(1-x^2)((1+x)^\alpha + (1-x)^\alpha)}. \tag{1.5}$$

The quantile function is obtained through functional inversion of $T_\alpha(x)$. The quantile function of the Yun-G family is given by;

$$T_\alpha^{-1}(y) = \frac{(1+y)^{1/\alpha} - (1-y)^{1/\alpha}}{(1+y)^{1/\alpha} + (1-y)^{1/\alpha}}. \tag{1.6}$$

The focus of this research lies in the under-explored potential of the Yun-G family of distributions. Hence, this study aims to extend the Fréchet distribution to broaden its utility in modeling failure data of components and maintenance data. Therefore, the present study proposed a versatile distribution from the Yun-G family called the Yun-Fréchet distribution (YFD). We also explored its general statistical properties and used five different methods of parameter estimation.

The paper is organized as follows: In Section 2 the model construction and basic statistical properties are discussed. Section 3 is devoted to characterizations of the distribution based on hazard function and truncated moments. In Section 4 five different methods of parametric estimation are studied. A simulation study has been done in Section 5. The flexibility and utility of the proposed model are studied in Section 6 and conclusions of this study are given in Section 7.

2 Yun Fréchet Distribution

In this section, we introduce a flexible distribution from the Yun-G family, based on Fréchet distribution.

Suppose X is a random variable that follows a YFD with parameters α, σ, and λ.
The distribution function of X is given by,

$$F(x) = \frac{\left(1+e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha} - \left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha}}{\left(1+e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha} + \left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha}}; \qquad x>0, \alpha \geq 1, \sigma > 0, \lambda > 0. \tag{2.1}$$

The corresponding pdf of X can be obtained as follows;

$$f(x) = 4\alpha\lambda\sigma^{\lambda} \frac{x^{-(\lambda+1)} e^{-\left(\frac{\sigma}{x}\right)^{\lambda}} \left(1-e^{-2\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha-1}}{\left(\left(1+e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha} + \left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha}\right)^{2}}; \qquad x>0, \alpha \geq 1, \sigma > 0, \lambda > 0. \tag{2.2}$$

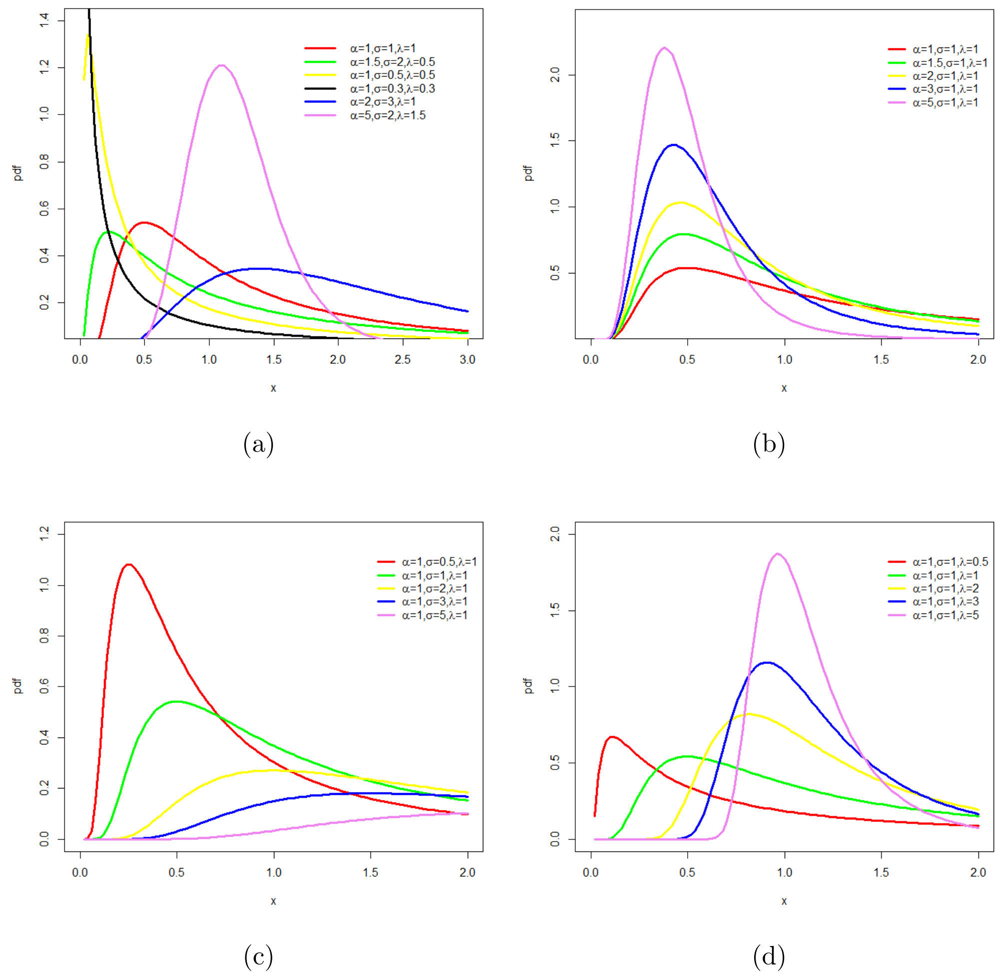
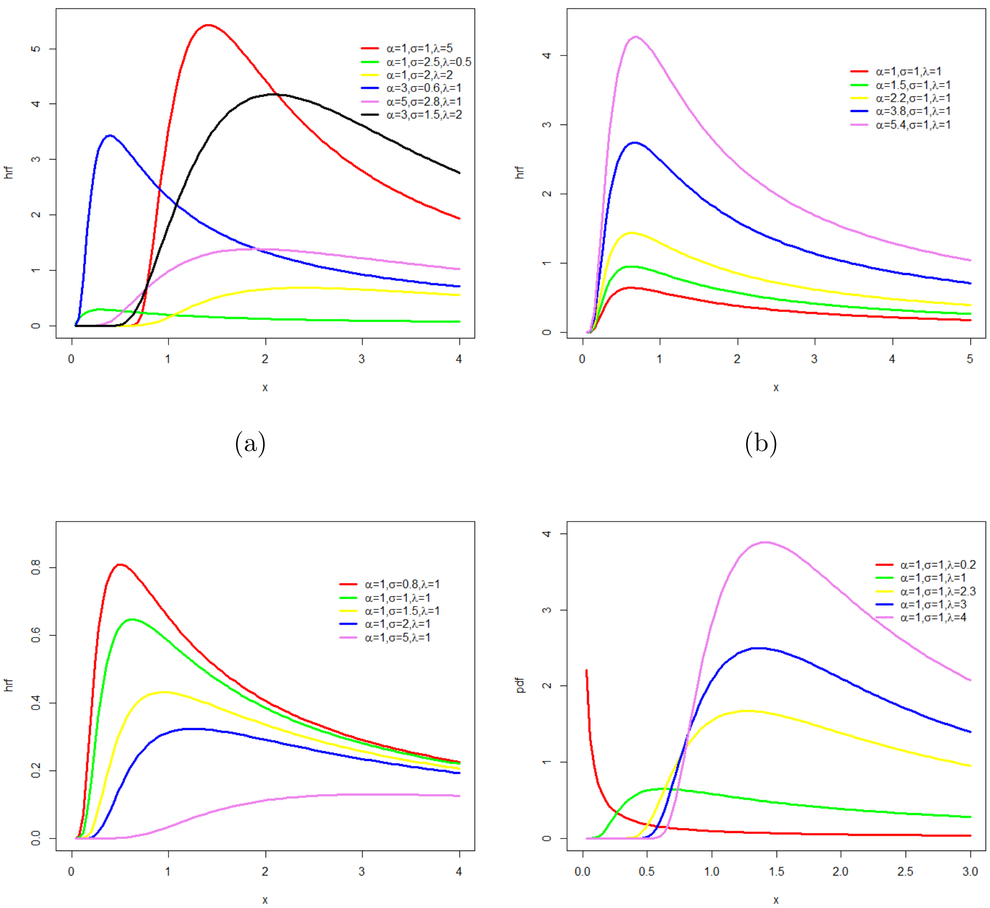


Figure 1: Plots of the pdf of the YFD for various parameter values.

From Figure 1 the pdf of YFD can be unimodal, increasing-decreasing, and right-skewed. The hrf of the YFD is given by;

$$h(x) = 2\alpha\lambda\sigma^{\lambda} \frac{x^{-(\lambda+1)} e^{-\left(\frac{\sigma}{x}\right)^{\lambda}} \left(1 - e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha}}{\left(1 - e^{-\left(\frac{\sigma}{x}\right)^{2\lambda}}\right) \left[\left(1 + e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha} + \left(1 - e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha}\right]}. \tag{2.3}$$



(a) (b)

(c) (d)

Figure 2: Plots of the hrf of the YFD for various parameter values.

Figure 2 shows the hazard function plots of YFD for different parameters and they show decreasing, increasing-decreasing, constant and unimodal behaviour.

The quantile function of the YFD is given by;

$$F^{-1}(y) = -\frac{\sigma}{\log\left[\frac{(1+y)^{1/\alpha}-(1-y)^{1/\alpha}}{(1+y)^{1/\alpha}+(1-y)^{1/\alpha}}\right]^{1/\lambda}}, \tag{2.4}$$

which will be used later for simulation purposes.

2.1 General properties of YFD

2.1.1 Moments

Moments are used to study the statistical characteristics of the distribution, such as mean, variance, skewness, and kurtosis. Moments of YFD random variable can be calculated as follows;

$$v_r' = E(X^r) = \int_0^\infty x^r \frac{4\alpha\lambda\sigma^\lambda x^{-(\lambda+1)} e^{-\left(\frac{\sigma}{x}\right)^\lambda} \left(1 - e^{-2\left(\frac{\sigma}{x}\right)^\lambda}\right)^{\alpha-1}}{\left(1 + e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right)^\alpha + \left(1 - e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right)^\alpha} dx \approx \sum_{k,\ell,m=0}^{K} b_{k,\ell,m} I_{\ell,m}(Q,G),$$

where K might be any large integer of choice, $b_{k,\ell,m} = a_k \binom{\alpha k}{\ell} \binom{-\alpha k}{m} (-1)^\ell (\ell + m), a_0 = 1$ and $a_k = 2(-1)^k$ for $k \geq 1$, and $I_{\ell,m}(Q,G)$ can be expressed as;

$$I_{\ell,m}(Q,G) = \int_0^\infty x^r \left[G(x)\right]^{\ell+m-1} g(x)\, dx = \lambda\sigma^\lambda \int_0^\infty x^r \left(e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right)^{\ell+m-1} x^{-(\lambda+1)} e^{-\left(\frac{\sigma}{x}\right)^\lambda}\, dx.$$

The mean, standard deviation, variance, skewness, and kurtosis for the YFD can be computed using the raw moments. With the help of R software, we computed them using the standard definitions.

The calculated values are presented in Table 1. It shows that the YFD is suitable for over-dispersed or under-dispersed data. The skewness and kurtosis values show positive skewness and leptokurtic behaviour. Also, the discussed moments for YFD decrease as the α value increases.

Table 1: Moment characteristics of the YFD for various parameter values.

Parameters	$\alpha \rightarrow$	2	4	5.5	6
	Mean	4.8777	3.6809	3.3694	3.2980
$\sigma = 5$	Variance	5.3667	0.4892	0.5565	177.3842
$\lambda = 2$	Skewness	4.9050	1.3473	0.8911	0.7980
	Kurtosis	743.8108	7.9976	5.2109	4.8035
	Mean	0.7782	0.5971	0.5487	0.5377
$\sigma = 0.8$	Variance	0.1195	0.0225	0.0133	0.0118
$\lambda = 2.1$	Skewness	4.2842	1.2782	0.8461	0.7569
	Kurtosis	204.8286	7.4824	5.0186	4.6500
	Mean	2.2291	1.7557	1.6274	1.5977
$\sigma = 2.3$	Variance	0.7712	0.1602	0.0972	0.0861
$\lambda = 2.3$	Skewness	3.4913	1.1639	0.7699	0.6870
	Kurtosis	64.1014	6.7145	4.7186	4.4092
	Mean	9.0623	8.2035	7.9457	7.8838
$\sigma = 9.3$	Variance	1.9334	0.6440	0.4409	0.4016
$\lambda = 5.2$	Skewness	1.4282	0.6039	0.3639	0.3086
	Kurtosis	8.0730	4.1919	3.6116	3.5104

2.1.2 Stochastic ordering

Stochastic ordering is a useful technique for comparing random variables in terms of statistical functions in distribution theory, see Shaked and Shanthikumar (2007) and Yaming (2009).

Let X_i be random variables with distribution functions $F_i(X)$ and density functions $f_i(x)$, the reliability functions be $\bar{F}_i(x)$, then we say that X_1 is smaller than X_2 if;

- $\bar{F}_1(x) \leq \bar{F}_2(x)$ for all $x, \Longrightarrow X_1 \leq_{st} X_2$ (Stochastic order).
- $\frac{f_1(x)}{\bar{F}_1(x)} \geq \frac{f_2(x)}{\bar{F}_2(x)}$ for all $x, \Longrightarrow X_1 \leq_{hr} X_2$ (Hazard rate order).
- $\frac{f_1(x)}{\bar{F}_1(x)} \geq \frac{f_2(x)}{\bar{F}_2(x)}$ for all $x, \Longrightarrow X_1 \leq_{rh} X_2$ (Reversed hazard rate order).
- $\frac{f_1(x)}{f_2(x)}$ is a monotonic decreasing function for all x,$\Longrightarrow X_1 \leq_{lr} X_2$ (Likelihood ratio

order).

Also, we have $X_1 \leq_{lr} X_2$ implies $X_1 \leq_{hr} X_2$ and $X_1 \leq_{rh} X_2$ which implies $X_1 \leq_{st} X_2$. Suppose the densities of X_1 and X_2 are, respectively,

$$f_1(x) = 4\alpha_1\lambda\sigma^{\lambda}\frac{x^{-(\lambda+1)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\left(1-e^{-2\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha_1-1}}{\left(\left(1+e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha_1}+\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha_1}\right)^2}; \qquad x>0, \text{and}$$

$$f_2(x) = 4\alpha_2\lambda\sigma^{\lambda}\frac{x^{-(\lambda+1)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\left(1-e^{-2\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha_2-1}}{\left(\left(1+e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha_2}+\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha_2}\right)^2}; \qquad x>0.$$

Then,

$$\frac{f_1(x)}{f_2(x)} = \frac{\alpha_1}{\alpha_2}\frac{\left(\left(1+e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha_2}+\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha_2}\right)^2\left(1-e^{-2\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha_1-\alpha_2}}{\left(\left(1+e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha_1}+\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha_1}\right)^2}.$$

For $\alpha_1 > \alpha_2$, $\left(\frac{f_1(x)}{f_2(x)}\right)' < 0$, which satisfies $X_1 \leq_{lr} X_2$.

2.1.3 Entropy

It is a common fact that every statistical distribution has a lack of certainty. Here, entropy acts as a measure of uncertainty. The concept of entropy was introduced by Shannon (1948). If X is a non-negative continuous random variable that has pdf $f(x)$, and cdf $F(x)$ then Shannon entropy of X is defined as;

$$S(X) = -\int_0^{\infty} f(x)\log(f(x))dx.$$

The Rényi Entropy Rényi (1961) is defined by,

$$H_\theta(X) = \frac{1}{1-\theta}\ln\left(\int_0^{\infty} f(x)^{\theta}dx\right); \qquad \theta > 0(\neq 1).$$

Using Eq.2.2 in the above expression, the Rényi entorpy of YFD can be written as;

$$H_\theta(X) = \frac{1}{1-\theta}\log\left(\int_0^\infty \left[\frac{4\alpha\lambda\sigma^\lambda x^{-(\lambda+1)}e^{-\left(\frac{\sigma}{x}\right)^\lambda}\left(1-e^{-2\left(\frac{\sigma}{x}\right)^\lambda}\right)^{(\alpha-1)}}{\left(\left(1+e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right)^\alpha+\left(1-e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right)^\alpha\right)^2}\right]^\theta dx\right)$$

$$= \frac{1}{1-\theta}\log(4\alpha\lambda\sigma^\lambda)^\theta + \frac{1}{1-\theta}\log\left(\int_0^\infty \left[\frac{x^{-(\lambda+1)}e^{-\left(\frac{\sigma}{x}\right)^\lambda}\left(1-e^{-2\left(\frac{\sigma}{x}\right)^\lambda}\right)^{(\alpha-1)}}{\left(\left(1+e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right)^\alpha+\left(1-e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right)^\alpha\right)^2}\right]^\theta dx\right)$$

$$= \frac{\theta}{1-\theta}\log(4) + \frac{\theta}{1-\theta}\log(\alpha) + \frac{\theta}{1-\theta}\log(\lambda) + \frac{\theta\lambda}{1-\theta}\log(\sigma) + \frac{1}{1-\theta}\log(U),$$

where

$$U = \int_0^\infty \frac{x^{-\theta(\lambda+1)}e^{-\theta\left(\frac{\sigma}{x}\right)^\lambda}\left(1-e^{-2\left(\frac{\sigma}{x}\right)^\lambda}\right)^{\theta(\alpha-1)}}{\left(\left(1+e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right)^\alpha+\left(1-e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right)^\alpha\right)^{2\theta}}dx.$$

Proposition 2.1. *We have*

$$U = \sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{m=0}^\infty\sum_{n=0}^\infty \binom{\theta(\alpha-1)}{k}\binom{-2\theta}{l}\binom{-\alpha(2\theta+k)}{m}\binom{-\alpha k}{n}$$
$$(-1)^k\int_0^\infty x^{-(\theta+1)}e^{-(2\theta(\alpha-1)+m+n)\left(\frac{\sigma}{x}\right)^\lambda}.$$

Proof. Using the binomial expansions successively,

$$x^{-\theta(\lambda+1)}e^{-\theta\left(\frac{\sigma}{x}\right)^\lambda}\left(1-e^{-2\left(\frac{\sigma}{x}\right)^\lambda}\right)^{\theta(\alpha-1)}\left(\left(1+e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right)^\alpha+\left(1-e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right)^\alpha\right)^{-2\theta}$$

$$= \sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{m=0}^\infty\sum_{n=0}^\infty \binom{\theta(\alpha-1)}{k}\binom{-2\theta}{l}\binom{-\alpha(2\theta+k)}{m}\binom{-\alpha k}{n}$$
$$(-1)^k x^{-(\theta+1)}e^{-(2\theta(\alpha-1)+m+n)\left(\frac{\sigma}{x}\right)^\lambda}.$$

Therefore,

$$U = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \binom{\theta(\alpha-1)}{k}\binom{-2\theta}{l}\binom{-\alpha(2\theta+k)}{m}\binom{-\alpha k}{n}$$
$$(-1)^k \int_0^{\infty} x^{-(\theta+1)} e^{-(2\theta(\alpha-1)+m+n)\left(\frac{\sigma}{x}\right)^{\lambda}}.$$

This ends the proof of Proposition 2.1. □

3 Characterizations of YFD

Understanding and accurately characterizing probability distributions is important in various fields, enabling insights into diverse phenomena. In this section, we established the characterizations of YFD based on hazard function and truncated moments. Over the years, many researchers have studied the different techniques of characterization of continuous probability distributions such as Glänzel (1987, 1988) and Hamedani (1993).

3.1 Characterization based on hazard function

The hrf $h(x)$ of a twice differentiable distribution function $F(x)$ and pdf $f(x)$ satisfies the first-order differential equation,

$$\frac{f'(x)}{f(x)} = \frac{h'(x)}{h(x)} - h(x). \tag{3.1}$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. Hamedani and Ahsanullah (2005) characterized certain widely recognized distributions based on the hazard function. The following characterization establishes a non-trivial characterization of YFD, when $\alpha = 1$, which is not of the above trivial form.

Theorem 3.1. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The pdf of X is Eq.2.2 if and only if its hazard function $h(x)$ satisfies the differential equation*

$$x^{\lambda+1}h'(x) + (\lambda+1)x^{\lambda}h(x) = \frac{d}{dx}\left[\frac{\lambda\sigma^{\lambda}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}\right]. \tag{3.2}$$

Proof. When $\alpha = 1$, the pdf $f(x)$ and hrf $h(x)$ of X are respectively

$$f(x) = \lambda\sigma^{\lambda}x^{-(\lambda+1)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}, \tag{3.3}$$

and

$$h(x) = \frac{\lambda\sigma^{\lambda}x^{-(\lambda+1)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}. \tag{3.4}$$

Then we have

$$\frac{f'(x)}{f(x)} = -\frac{(\lambda+1)}{x} + \lambda\sigma^{\lambda}x^{-(\lambda+1)}. \tag{3.5}$$

Using Eq.3.1 we can write,

$$h'(x) + h(x)\frac{(\lambda+1)}{x} = -\frac{\lambda^2\sigma^{2\lambda}x^{-2(\lambda+1)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^2},$$

which implies,

$$x^{\lambda+1}h'(x) + (\lambda+1)x^{\lambda}h(x) = -\frac{\lambda^2\sigma^{2\lambda}x^{-2(\lambda+1)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^2}.$$

Now, Eq.3.2 holds, then

$$\frac{d}{dx}\left(x^{\lambda+1}h(x)\right) = \frac{d}{dx}\left(\frac{\lambda\sigma^{\lambda}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}\right)$$

from which we obtain

$$h(x) = \frac{\lambda\sigma^{\lambda}x^{\lambda+1}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}},$$

which is the hrf of YFD when $\alpha = 1$. $\square$

3.2 Characterization based on truncated moments

The exploration of characterizing probability distributions through truncated moments originated from the research by Galambos and Kotz (1978). Subsequent advancements in this area were made by various scholars and researchers. Notable contributors to the development of characterizations using truncated moments include Kotz and Shanbag (1980), Glänzel et al. (1984) and Glänzel (1987,1990). The YFD is characterized through truncated moments. This is developed based on the Theorem 3.2 of Glänzel (1987), which is stated as follows,

Theorem 3.2. *Let (Ω, Σ, P) be a given probability space, and let $H = [a, b]$ be an interval for some $a < b$ ($a = \infty, b = -\infty$ might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with distribution function $G(x)$ and let q_1 and q_2 be two real functions defined on H such that*

$$E[q_1(X)|X \geq x] = E[q_2(X)|X \geq x]\eta(x), x \in H$$

is defined with some real function η. Assume that $q_1, q_2 \in C_1(H), \eta \in C_2(H)$, and $G(x)$ is a twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $q_2\eta = q_1$ has no real solution in the interior of H. Then G is uniquely determined by the functions q_1, q_2 and η. In particular,

$$G(x) = \int_a xC \left| \frac{\eta^{'}(u)}{\eta(u)q_2(u) - q_1(u)} \right| e^{-s(u)}$$

where the function s is a solution of the differential equation $s' = \frac{\eta' q_2}{\eta q_2 - q_1}$ and C is a constant chosen to make $\int_H dG = 1$.

The above theorem has the advantage that the cdf G is not required to have a closed form and is given in terms of an integral whose integrand depends on the solution of a first-order differential equation, which can serve as a bridge between probability and differential equation.

Theorem 3.3. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable, and let*
$q_2(x) = \left(\left(1 + e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right)^\alpha + \left(1 - e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right)^\alpha\right)^2 \left(1 - e^{-2\left(\frac{\sigma}{x}\right)^\lambda}\right)^{\alpha-1}$ *and* $q_1(x) = q_2(x)e^{-\left(\frac{\sigma}{x}\right)^\lambda}$
for $x > 0$. The pdf of X is Eq.2.2 if and only if the function η defined in Theorem 3.2 has the form

$$\eta(x)=\frac{1}{2}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}},x>0. \tag{3.6}$$

Proof. Let X have pdf Eq.2.2, then

$$(1-G(x))E[q_2(X)|X\geq x]=8\alpha\lambda\sigma^{\lambda}\frac{x^{-(\lambda+1)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha}}{\left(1+e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha}-\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha}}, \qquad x>0,$$

$$(1-G(x))E[q_2(X)|X\geq x]=8\alpha\lambda\sigma^{\lambda}\frac{x^{-(\lambda+1)}e^{-2\left(\frac{\sigma}{x}\right)^{\lambda}}\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha}}{\left(1+e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha}-\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha}}, \qquad x>0,$$

and then

$$\eta(x)q_2(x)-q_1(x)=-\frac{1}{2}\left(\left(1+e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha}+\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha}\right)^{2}\left(1-e^{-2\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha-1}<0, \quad for \;\; x>0.$$

Conversely, if η is given as Eq.3.6, then

$$s'(x)=\frac{\eta'(x)q_2(x)}{\eta(x)q_2(x)-q_1(x)}=\lambda\sigma^{\lambda}x^{-(\lambda+1)},x>0,$$

and hence,

$$s(x)=\left(\frac{\sigma}{x}\right)^{\lambda}$$

or

$$e^{-s(x)}=e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}.$$

Now, using Theorem 3.2, X has the pdf Eq.2.2. □

4 Estimation Methods

In this section we describe five methods of estimation, such as maximum likelihood method, ordinary least square method, weighted least square method, Cramér-Von Mises method, and Anderson-Darling method for estimating the parameters α, σ and λ of YFD.

4.1 Method of maximum Likelihood Estimation

It is well-known that the method of maximum likelihood is the most popular method in statistical inference Casella and Berger (2002), and its significance because it includes the probabilistic structure that the considered distribution's pdf represents. Let $x_1, x_2, \ldots, x_n$ are the observed values of a random sample from the YFD. The log-likelihood for $\theta = (\alpha, \sigma, \lambda)^T$ based on the mentioned values is given by;

$$\begin{aligned}\log L(\alpha, \lambda, \sigma) =& n\log(4) + n\log(\alpha) + n\log(\lambda) + n\lambda\log(\sigma)\\ &-(\lambda+1)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\left(\frac{\sigma}{x_i}\right)^{\lambda} + (\alpha-1)\sum_{i=1}^{n}\log\left(1-e^{-2\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)\\ &-2\sum_{i=1}^{n}\log\left[\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha} + \left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}\right].\end{aligned}$$

The partial derivatives of this function with respect to α, λ and σ are given by;

$$\begin{aligned}\frac{\partial \log L(\alpha,\lambda,\sigma)}{\partial\alpha} =& \left(\frac{n}{\alpha}\right) + \sum_{i=1}^{n}\log\left(1-e^{-2\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right) - \\ & 2\sum_{i=1}^{n}\frac{\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}\log\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right) + \left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}\log\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)}{\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha} + \left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}},\end{aligned} \tag{4.1}$$

$$\begin{aligned}\frac{\partial \log L(\alpha,\lambda,\sigma)}{\partial\sigma} =& \frac{n\lambda}{\sigma} - \lambda\sigma^{\lambda-1}\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\lambda} + 2(\alpha-1)\lambda\sigma^{\lambda-1}\sum_{i=1}^{n}\left(\frac{e^{-2\left(\frac{\sigma}{x_i}\right)^{\lambda}}\left(\frac{1}{x_i}\right)^{\lambda}}{1-e^{-2\left(\frac{\sigma}{x_i}\right)^{\lambda}}}\right) + 2\alpha\lambda\sigma^{\lambda-1}\\ & \sum_{i=1}^{n}\frac{\left(\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha-1} - \left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha-1}\right)\left(e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\left(\frac{1}{x_i}\right)^{\lambda}\right)}{\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha} + \left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}},\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\frac{\partial \log L(\alpha,\lambda,\sigma)}{\partial\lambda} =& \left(\frac{n}{\lambda}\right) - n\log(\sigma) - \sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\left(\frac{\sigma}{x_i}\right)^{\lambda}\log\left(\frac{\sigma}{x_i}\right) - \\ & 2(\alpha-1)\sum_{i=1}^{n}\frac{\left(\frac{\sigma}{x_i}\right)^{\lambda}\log\left(\frac{\sigma}{x_i}\right)e^{-2\left(\frac{\sigma}{x_i}\right)^{\lambda}}}{1-e^{-2\left(\frac{\sigma}{x_i}\right)^{\lambda}}} + 2\alpha\sigma^{\lambda}\sum_{i=1}^{n}e^{\left(\frac{\sigma}{x_i}\right)^{\lambda}}\left(\frac{1}{x_i}\right)^{\lambda}\\ & \log\left(\frac{\sigma}{x_i}\right)\frac{\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha-1} - \left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha-1}}{\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha} + \left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}}.\end{aligned} \tag{4.3}$$

Maximum Likelihood Estimate (MLE) $\hat{\theta} = (\hat{\alpha}, \hat{\sigma}, \hat{\lambda})^T$ of $\theta = (\alpha, \sigma, \lambda)^T$ can be obtained by solving simultaneously the following normal equations.

$$\frac{\partial \log L(\alpha, \lambda, \sigma)}{\partial \alpha} = 0; \qquad \frac{\partial \log L(\alpha, \lambda, \sigma)}{\partial \sigma} = 0; \qquad \frac{\partial \log L(\alpha, \lambda, \sigma)}{\partial \lambda} = 0.$$

We propose other methods of estimation to the distribution parameters and assess their performance with the MLE method via simulation.

Theorem 4.1. *From Eq.4.1, let $f_1(\alpha, \sigma, \lambda) = \partial \log L(\alpha, \lambda, \sigma)/\partial \lambda$. Then, there exists a solution for $f_1(\alpha, \sigma, \lambda) = 0$ for $\alpha \geq 1$ and the solution is unique when $-n/2\alpha^2 < S_1$, where*

$$S_1 = 2\sum_{i=1}^{n} \frac{\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}\left[log\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)-log\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)\right]^2}{\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}+\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}}.$$

Proof. We have

$$\begin{aligned} f_1(\alpha, \sigma, \lambda) = &\left(\frac{n}{\alpha}\right) + \sum_{i=1}^{n} \log\left(1 - e^{-2\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right) - \\ & 2\sum_{i=1}^{n} \frac{\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}\log\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)+\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}\log\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)}{\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}+\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}}, \end{aligned}$$

Since σ and λ are given, then the limiting values of $f_1(\alpha, \sigma, \lambda)$ are

$$\lim_{\alpha\to\infty} f_1(\alpha, \sigma, \lambda) = \sum_{i=1}^{n} \log\left(1 - e^{-2\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right) - 2\sum_{i=1}^{n} \log\left(1 + e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right) < 0,$$

and

$$\begin{aligned} \lim_{\alpha\to 1} f_1(\alpha, \sigma, \lambda) =& n + \sum_{i=1}^{n} \log\left(1 - e^{-2\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right) - \\ & \sum_{i=1}^{n} \left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}\log\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)+\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}\log\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right), \end{aligned}$$

and we can show that it approaches a positive value numerically. Thus there exists at least one root say $\hat{\alpha} \in (1, \infty)$.

To show uniqueness, we have to show that $\partial \log L(\alpha,\lambda,\sigma)/\partial\alpha < 0$, that is,

$$-\frac{n}{\alpha^2} - 2\sum_{i=1}^{n} \frac{\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^\lambda}\right)^\alpha \left(1+e^{-\left(\frac{\sigma}{x_i}\right)^\lambda}\right)\left[\log\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^\lambda}\right) - \log\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^\lambda}\right)\right]}{\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^\lambda}\right)^\alpha + \left(1-e^{-\left(\frac{\sigma}{x_i}\right)^\lambda}\right)^\alpha} < 0,$$

that is, $-\frac{n}{2\alpha^2} < S_1$. Hence, there exists a solution for $f_1(\alpha,\sigma,\lambda) = 0$, and the root is unique when $-\frac{n}{2\alpha^2} < S_1$. □

Theorem 4.2. *From Eq.4.2, let $f_2(\alpha,\sigma,\lambda) = \partial\, log\, L(\alpha,\lambda,\sigma)/\partial\sigma$. Then, there exists a solution for $f_2(\alpha,\sigma,\lambda) = 0$ for $\sigma \in (0,\infty)$ and the solution is unique when $\lambda^2 S_2 + S_3 > 2\lambda^2\alpha S_4$.*

Proof. We have

$$\begin{aligned} f_2(\alpha,\sigma,\lambda) =& n\lambda - \lambda\sigma^\lambda \sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^\lambda + 2(\alpha-1)\lambda\sigma^\lambda \sum_{i=1}^{n}\left(\frac{e^{-2\left(\frac{\sigma}{x_i}\right)^\lambda}\left(\frac{1}{x_i}\right)^\lambda}{1-e^{-2\left(\frac{\sigma}{x_i}\right)^\lambda}}\right) + 2\alpha\lambda\sigma^\lambda \\ & \sum_{i=1}^{n} \frac{\left(\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^\lambda}\right)^{\alpha-1} - \left(1+e^{-\left(\frac{\sigma}{x_i}\right)^\lambda}\right)^{\alpha-1}\right)\left(e^{-\left(\frac{\sigma}{x_i}\right)^\lambda}\left(\frac{1}{x_i}\right)^\lambda\right)}{\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^\lambda}\right)^\alpha + \left(1-e^{-\left(\frac{\sigma}{x_i}\right)^\lambda}\right)^\alpha}. \end{aligned}$$

Since α and λ are given, then the limiting values of $f_2(\alpha,\sigma,\lambda)$ are

$$\lim_{\sigma\to\infty} f_2(\alpha,\sigma,\lambda) = -\infty,$$

and

$$\lim_{\sigma\to 0} f_2(\alpha,\sigma,\lambda) > 0,$$

and we can show that it approaches a positive value numerically. Thus, at least one root say $\hat{\sigma} \in (0,\infty)$ exists. To show uniqueness, we have to show that $\partial \log L(\alpha,\lambda,\sigma)/\partial\sigma < 0$, that is,

$$-\frac{\lambda^2}{\sigma}\sum_{i=1}^{n} m-\frac{2(\alpha-1)\lambda}{\sigma}\sum_{i=1}^{n} e^{-2m} m\frac{\left(2m-m+e^{-2m}\right)}{\left(e^{-2m}-1\right)^2}+\frac{2\lambda^2\alpha}{\sigma}\sum_{i=1}^{n} e^{-m}\left[\frac{(m-m^2)v^{2(\alpha-1)}}{\left(u^\alpha+v^\alpha\right)^2}-\right.$$
$$\frac{m^2e^{-m}v^{2(\alpha-2)}+u^{\alpha-1}v^{\alpha-1}\left(2\alpha m^2e^{-m}+u(m-m^2)\right)+m^2e^{-m}u^{2(\alpha-2)}}{\left(u^\alpha+v^\alpha\right)^2}+$$
$$\left.\frac{\left((m^2-m)u^{\alpha-1}+(\alpha-1)m^2e^{-m}u^{\alpha-2}\right)v^\alpha}{\left(u^\alpha+v^\alpha\right)^2}\right]<0,$$

where, $m=\left(\frac{\sigma}{x_i}\right)^\lambda$, $u=1+e^{-(\frac{\sigma}{x_i})^\lambda}$ and $v=1-e^{-(\frac{\sigma}{x_i})^\lambda}$.

i.e.,

$$\lambda^2\sum_{i=1}^{n} m+2(\alpha-1)\lambda\sum_{i=1}^{n} e^{-2m} m\frac{\left(2m-m+e^{-2m}\right)}{\left(e^{-2m}-1\right)^2}-2\lambda^2\alpha\sum_{i=1}^{n} e^{-m}\left[\frac{(m-m^2)v^{2(\alpha-1)}}{\left(u^\alpha+v^\alpha\right)^2}-\right.$$
$$\frac{m^2e^{-m}v^{2(\alpha-2)}+u^{\alpha-1}v^{\alpha-1}\left(2\alpha m^2e^{-m}+u(m-m^2)\right)+m^2e^{-m}u^{2(\alpha-2)}}{\left(u^\alpha+v^\alpha\right)^2}+$$
$$\left.\frac{\left((m^2-m)u^{\alpha-1}+(\alpha-1)m^2e^{-m}u^{\alpha-2}\right)v^\alpha}{\left(u^\alpha+v^\alpha\right)^2}\right]>0,$$

$$\lambda^2S_2+S_3>2\lambda^2\alpha S_4,$$

Hence, there exists a solution for $f_2(\alpha,\sigma,\lambda)=0$, and the root is unique when $\lambda^2S_2+S_3>2\lambda^2\alpha S_4$. □

Theorem 4.3. *From Eq.4.3, let $f_3(\alpha,\sigma,\lambda)=\partial\log L(\alpha,\lambda,\sigma)/\partial\lambda$. Then, there exists a solution for $f_3(\alpha,\sigma,\lambda)=0$, and the solution is unique when $\frac{n}{\lambda^2}+S_5>2(\alpha-1)S_6+2\alpha S_7$.*

Proof. We have

$$f_3(\alpha,\sigma,\lambda)=\left(\frac{n}{\lambda}\right)-n\log(\sigma)-\sum_{i=1}^{n}\log(x_i)-\sum_{i=1}^{n}\left(\frac{\sigma}{x_i}\right)^\lambda\log\left(\frac{\sigma}{x_i}\right)-$$
$$2(\alpha-1)\sum_{i=1}^{n}\frac{\left(\frac{\sigma}{x_i}\right)^\lambda\log\left(\frac{\sigma}{x_i}\right)e^{-2\left(\frac{\sigma}{x_i}\right)^\lambda}}{1-e^{-2\left(\frac{\sigma}{x_i}\right)^\lambda}}+2\alpha\sigma^\lambda\sum_{i=1}^{n} e^{\left(\frac{\sigma}{x_i}\right)^\lambda}\left(\frac{1}{x_i}\right)^\lambda$$
$$\log\left(\frac{\sigma}{x_i}\right)\frac{\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^\lambda}\right)^{\alpha-1}-\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^\lambda}\right)^{\alpha-1}}{\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^\lambda}\right)^{\alpha}+\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^\lambda}\right)^{\alpha}}.$$

The limiting values of $f_3(\alpha,\sigma,\lambda)$ as $\lambda \longrightarrow 0$ and $\lambda \longrightarrow \infty$ are, respectively, $\lim_{\lambda\to\infty} f_3(\alpha,\sigma,\lambda) = -\infty$ and $\lim_{\lambda\to 0} f_3(\alpha,\sigma,\lambda) = \infty$. Thus, there exists at least one root, say $\hat{\lambda} \in (0,\infty)$. To show uniqueness, we have to show that $\partial \log L(\alpha,\lambda,\sigma)/\partial\lambda < 0$, that is,

$$
\begin{aligned}
&-\frac{n}{\lambda^2} - \sum_{i=1}^{n} m\log\left(\frac{\sigma}{x_i}\right)^2 \\
&+2(\alpha-1)\sum_{i=1}^{n} \frac{\left[2\lambda m^2\log\left(\frac{\sigma}{x_i}\right) + m\left(\log(m)+1\right)\left(e^{-2m}-1\right)\right]\left(e^{-2m}\log\left(\frac{\sigma}{x_i}\right)\right)}{\left(e^{-2m}-1\right)^2} \\
&+2\alpha\sum_{i=1}^{n} e^{-m}\left(\log\left(\frac{\sigma}{x_i}\right)\right)^2\left[\frac{\left(m-m^2\right)v^{2(\alpha-1)} - e^{-m}v^{2(\alpha-2)}m^2 + \left(2\alpha e^{-m}u^{a-1}m^2 + u^{\alpha}\left(m-m^2\right)\right)v^{\alpha-1}}{\left(u^{\alpha}+v^{\alpha}\right)^2}\right. \\
&\left.+\frac{\left(-e^{-m}m - m + m^2\right)u^{2(\alpha-2)} + e^{-m}u^{\alpha}m^2v^{\alpha-2}(\alpha-1) + \left(u^{\alpha-2}\left(m^2\alpha - m\right)e^{-m} - m + m^2\right)v^{\alpha}}{\left(u^{\alpha}+v^{\alpha}\right)^2}\right] < 0,
\end{aligned}
$$

where, $m = \left(\frac{\sigma}{x_i}\right)^{\lambda}$, $u = 1 + e^{-(\frac{\sigma}{x_i})^{\lambda}}$ and $v = 1 - e^{-(\frac{\sigma}{x_i})^{\lambda}}$,
i.e.,

$$
\begin{aligned}
&\frac{n}{\lambda^2} + \sum_{i=1}^{n} m\log\left(\frac{\sigma}{x_i}\right)^2 \\
&-2(\alpha-1)\sum_{i=1}^{n} \frac{\left[2\lambda m^2\log\left(\frac{\sigma}{x_i}\right) + m\left(\log(m)+1\right)\left(e^{-2m}-1\right)\right]\left(e^{-2m}\log\left(\frac{\sigma}{x_i}\right)\right)}{\left(e^{-2m}-1\right)^2} \\
&-2\alpha\sum_{i=1}^{n} e^{-m}\left(\log\left(\frac{\sigma}{x_i}\right)\right)^2\left[\frac{\left(m-m^2\right)v^{2(\alpha-1)} - e^{-m}v^{2(\alpha-2)}m^2 + \left(2\alpha e^{-m}u^{a-1}m^2 + u^{\alpha}\left(m-m^2\right)\right)v^{\alpha-1}}{\left(u^{\alpha}+v^{\alpha}\right)^2}\right. \\
&\left.+\frac{\left(-e^{-m}m - m + m^2\right)u^{2(\alpha-2)} + e^{-m}u^{\alpha}m^2v^{\alpha-2}(\alpha-1) + \left(u^{\alpha-2}\left(m^2\alpha - m\right)e^{-m} - m + m^2\right)v^{\alpha}}{\left(u^{\alpha}+v^{\alpha}\right)^2}\right] > 0,
\end{aligned}
$$

$$
\frac{n}{\lambda^2} + S_5 > 2(\alpha-1)S_6 + 2\alpha S_7,
$$

Hence, there exists a solution for $f_3(\alpha,\sigma,\lambda) = 0$, and the root is unique when $\frac{n}{\lambda^2} + S_5 > 2(\alpha-1)S_6 + 2\alpha S_7$. □

4.2 Method of Ordinary least squares

This method of estimation was introduced by James et al. (1988). The ordinary least squares (OLS) estimators can be obtained by minimizing

$$
\sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{i}{n+1}\right]^2
$$

with respect to α, σ, and λ.

That is $\hat{\theta} = (\hat{\alpha}, \hat{\sigma}, \hat{\lambda})^T$ can be obtained by minimizing;

$$\sum_{i=1}^{n}\left[\frac{\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}-\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}}{\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}+\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}}-\frac{i}{n+1}\right]^2$$

with respect to α, σ, and λ.

4.3 Method of Weighted least squares

The weighted least square (WLS) estimator of YFD parameters are obtained by minimizing

$$\sum_{i=1}^{n} v_i\left[F(x_{i:n})-\frac{i}{n+1}\right]^2$$

with respect to α, σ, and λ where $v_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}$ are the weights.
Hence,

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{(n-i+1)}\left[\frac{\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}-\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}}{\left(1+e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}+\left(1-e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}\right)^{\alpha}}-\frac{i}{n+1}\right]^2$$

by minimizing this equation with respect to α, σ, and λ, we obtain $\hat{\alpha}_{WLS}, \hat{\sigma}_{WLS}$, and $\hat{\lambda}_{WLS}$.

4.4 Cramér-Von Mises method

This is another approach for estimating the parameters proposed by Macdonald (2018). Cramér-Von Mises (CVM) estimators are also known as minimum distance estimators. The CVM estimators are obtained by minimizing,

$$c(\alpha, \sigma, \lambda) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_i) - \frac{2i-1}{2n}\right]^2.$$

These estimators can also be obtained by solving the following non-linear equations.

$$\sum_{i=1}^{n}\left[F(x_i)-\frac{2i-1}{2n}\right]\Delta_1(x_i|\alpha,\sigma,\lambda)=0,$$
$$\sum_{i=1}^{n}\left[F(x_i)-\frac{2i-1}{2n}\right]\Delta_2(x_i|\alpha,\sigma,\lambda)=0,$$
$$\sum_{i=1}^{n}\left[F(x_i)-\frac{2i-1}{2n}\right]\Delta_3(x_i|\alpha,\sigma,\lambda)=0,$$

where

$$\Delta_1(x_{i:n}|\alpha,\sigma,\lambda)=\frac{2\left(1-e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)^\alpha\left(1+e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)^\alpha\left(ln\left(1+e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)-ln\left(1-e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)\right)}{\left[\left(1+e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)^\alpha+\left(1-e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)^\alpha\right]^2},$$

$$\Delta_2(x_{i:n}|\alpha,\sigma,\lambda)=-\frac{4\alpha\lambda(\frac{\sigma}{x_{i:n}})^\lambda\left(1-e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)^\alpha\left(1+e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)^\alpha e^{\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}}{\left(\left(1+e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)^\alpha+\left(1-e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)^\alpha\right)^2\left(1+e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)+\left(1-e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)},$$

$$\Delta_3(x_{i:n}|\alpha,\sigma,\lambda)=-\frac{4\alpha ln\left(\frac{\sigma}{x_{i:n}}\right)(\frac{\sigma}{x_{i:n}})^\lambda\left(1-e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)^\alpha\left(1+e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)^\alpha e^{\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}}{\left(\left(1+e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)^\alpha+\left(1-e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)^\alpha\right)^2\left(1+e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)+\left(1-e^{-\left(\frac{\sigma}{x_{i:n}}\right)^\lambda}\right)}.$$

4.5 Anderson-Darling method

The Anderson-Darling test was introduced by Anderson and Darling Anderson and Darling (1952). The estimators are obtained by minimizing

$$A(\alpha,\sigma,\lambda)=-n-\frac{1}{n}+\sum_{i=1}^{n}(2i-1)\left[\log F\left(x_{1:n}|\alpha,\sigma,\lambda\right)+\log\bar{F}\left(x_{1:n}|\alpha,\sigma,\lambda\right)\right].$$

The Anderson-Darling (AD) estimates of YFD parameters can also be obtained by solving the following non-linear equations.

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_1(x_i|\alpha,\sigma,\lambda)}{F(x_i|\alpha,\sigma,\lambda)}-\frac{\Delta_1(x_{n+1-1:n}|\alpha,\sigma,\lambda)}{\bar{F}(x_{n+1-1:n}|\alpha,\sigma,\lambda)}\right]=0,$$
$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_2(x_i|\alpha,\sigma,\lambda)}{F(x_i|\alpha,\sigma,\lambda)}-\frac{\Delta_2(x_{n+1-1:n}|\alpha,\sigma,\lambda)}{\bar{F}(x_{n+1-1:n}|\alpha,\sigma,\lambda)}\right]=0,$$
$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_3(x_i|\alpha,\sigma,\lambda)}{F(x_i|\alpha,\sigma,\lambda)}-\frac{\Delta_3(x_{n+1-1:n}|\alpha,\sigma,\lambda)}{\bar{F}(x_{n+1-1:n}|\alpha,\sigma,\lambda)}\right]=0,$$

where $\Delta_1(x_i|\alpha,\sigma,\lambda)$, $\Delta_2(x_i|\alpha,\sigma,\lambda)$, and $\Delta_3(x_i|\alpha,\sigma,\lambda)$ are discussed in Section 3.4.

5 Simulation

To determine the accuracy of parametric estimation, a Monte Carlo simulation is conducted in this section. Five different estimates of YFD parameters were examined in this simulation study. Using the quantile function of YFD given by Eq.2.4, we generated a random sample of observation for sizes $\boldsymbol{n = 50, 100, 200, 300,}$ and $\mathbf{500}$ with $\boldsymbol{N = 1000}$ replications. The two combinations of parameter values considered are; $\boldsymbol{\alpha = 5, \sigma = 2, \lambda = 1}$ and $\boldsymbol{\alpha = 1.2, \sigma = 2.5, \lambda = 0.2}$. The numerical outcomes are evaluated using R statistical Programming language with the widely used optimization package 'optim'. The Average Value, Mean Square Error (MSE), and Average Bias have been computed and displayed in Table 2 and Table 3. The values show that as sample size increases the MSE decreases and the Average Value of each parameter converges to initial parameter values. These findings demonstrate the accuracy and consistency of the estimation techniques.

6 Fitting and applicability to data

In this section, we present two real world applications to showcase the practicality and versatility of the proposed model. The performance of YFD was compared to that of Yun Weibull distribution (YWD), Yun Exponentiated distribution (YED), which are submodels of Yun-G family that are discussed in Chesneau et al. (2021), the Transmuted Fréchet distribution (TFD) by Mahmoud and Mandouh (2013) and Exponentiated Weibull distribution (EWD) by Pal et al. (2006). The maximum likelihood method is employed to estimate the parameters for the candidate models. We evaluated different goodness-of-fit measures to illustrate the flexibility of the model. Specifically, $-logL$(negative log-likelihood function), W (Cramér-von Mises Statistic), A (Anderson-Darling Statistic) $K{-}S$ (Kolmogorov–Smirnov Statistic), AIC (Akaike Information Criterion), $CAIC$ (Akaike Information Criterion with correction), BIC (Bayesian Information Criterion) and $HQIC$ (Hannan–Quinn Information Criterion), where

Table 2: Simulation results.

True Value	**n**	**Average Value**					**Mean Square Error**					**Average Bias**				
		MLE	OLS	WLS	CVM	AD	MLE	OLS	WLS	CVM	AD	MLE	OLS	WLS	CVM	AD
$\alpha = 5$	50	5.5088	5.4012	12.1231	5.4029	5.8596	0.2588	0.2173	621.9664	0.216	0.7694	-0.5088	-0.4012	-7.1231	-0.4029	-0.8596
	100	5.3114	5.3237	7.8697	5.3527	5.8198	0.0988	0.1533	86.9136	0.1595	0.6966	-0.3144	-0.3237	-2.8697	-0.3527	-0.8198
	200	5.3194	5.3025	6.2406	5.2747	5.7656	0.102	0.1279	21.9848	0.1094	0.6167	-0.3194	-0.3025	-1.2406	-0.2747	-0.7656
	300	5.212	5.2638	5.7334	5.2201	5.7247	0.045	0.1114	9.7879	0.0912	0.5584	-0.212	-0.2638	-0.7334	-0.2201	-0.7247
	500	5.2044	5.2303	5.3672	5.2926	5.6643	0.0418	0.0978	3.8279	0.1234	0.4747	-0.2044	-0.2302	-0.3672	-0.2926	-0.6643
$\sigma = 2$	50	2.1595	2.1774	4.4209	1.9903	0.2738	0.0254	0.0962	42.0413	0.0674	2.9909	-0.1595	-0.1774	-2.4209	0.0097	1.7262
	100	2.095	2.1274	3.0814	2.0373	0.3159	0.009	0.0645	12.4571	0.0673	2.8476	-0.095	-0.1274	-1.0814	-0.0373	1.6841
	200	2.1031	2.182	2.4451	2.0702	0.3723	0.0106	0.0785	2.7839	0.0581	2.6624	-0.1031	-0.182	-0.4451	-0.0702	1.6277
	300	2.072	2.2143	2.2661	2.0997	0.406	0.0052	0.0904	1.1775	0.0619	2.554	-0.072	-0.2143	-0.2661	-0.0997	1.594
	500	2.0669	2.2019	2.1279	2.102	0.4512	0.0045	0.0792	0.4282	0.054	2.4097	-0.0669	-0.2019	-0.1279	-0.102	1.5488
$\lambda = 1$	50	1.0567	0.6924	1.1411	0.9044	2.4002	0.0032	0.2784	0.5738	0.0993	2.0369	-0.0567	0.3076	-0.1411	0.0956	-1.4002
	100	1.0431	0.7914	1.0482	1.014	2.4445	0.0019	0.1113	0.1153	0.0584	2.1428	-0.0431	0.2086	-0.0482	-0.014	-1.4445
	200	1.0238	0.8702	1.0268	1.0453	2.4805	0.0006	0.0669	0.0573	0.0589	2.2582	-0.0238	0.1298	-0.0268	-0.0453	-1.4805
	300	1.0198	0.8965	1.0155	1.0549	2.4891	0.0004	0.0587	0.0379	0.0564	2.2893	-0.0198	0.1035	-0.0155	-0.0549	-1.4891
	500	1.0083	0.9294	1.0115	1.0304	2.5426	0.0001	0.0538	0.0196	0.0417	2.4547	-0.0083	0.0706	-0.0115	-0.0304	-1.5426

Table 3: Simulation results.

True Value	n	Average Value					Mean Square Error					Average Bias				
		MLE	OLS	WLS	CVM	AD	MLE	OLS	WLS	CVM	AD	MLE	OLS	WLS	CVM	AD
$\alpha = 1.2$	50	71.2223	1.3681	1.4091	1.3811	1.7856	0.0005	0.0356	0.8175	0.0398	0.0369	-0.0223	-0.1681	-0.2091	-0.1811	-0.5856
	100	1.2182	1.3706	1.3285	1.3985	1.7057	0.0003	0.0359	0.449	0.0049	0.2795	-0.0182	-0.1706	-0.1285	0.0015	-0.5057
	200	1.2253	1.4021	1.2857	1.4173	1.6622	0.0006	0.0451	0.1934	0.0521	0.2429	-0.0253	-0.2021	-0.0857	-0.2173	-0.4622
	300	1.2216	1.4178	1.2428	1.4201	1.6173	0.0005	0.0516	0.1154	0.0531	0.2119	-0.0216	-0.2178	-0.0428	-0.2201	-0.4173
	500	1.2214	1.4332	1.2298	1.419	1.5786	0.0005	0.0573	0.0655	0.0518	0.1834	-0.0214	-0.2332	-0.0298	-0.219	-0.3786
$\sigma = 2.5$	50	3.1138	2.5422	620.5277	2.4953	1.9917	0.3768	0.013	28512490	0.0108	0.2786	-0.6138	-0.0422	-618.0277	0.0047	0.5083
	100	3.0994	2.5422	143.0358	2.489	2.0151	0.3592	0.0139	1286037	0.0082	0.2444	-0.5994	-0.0422	-140.5358	0.011	0.4849
	200	3.1021	2.5202	23.9949	2.4721	2.0214	0.3625	0.0085	13382.18	0.0073	0.2369	-0.6021	-0.0202	-21.4949	0.0279	0.4786
	300	3.0456	2.5174	9.1831	2.4661	2.018	0.2977	0.0069	1051.056	0.0067	0.2391	-0.5456	-0.0174	-6.6831	0.0339	0.482
	500	2.9697	2.5029	5.2306	2.4683	2.0295	0.2206	0.0053	131.776	0.0063	0.2281	-0.4697	-0.0029	-2.7306	0.0317	0.4705
$\lambda = 0.2$	50	0.2053	0.1593	0.2542	0.1895	1.1562	0.00002809	0.0034	0.0379	0.0032	0.9685	-0.0053	0.0407	-0.0542	0.0105	-0.9562
	100	0.2024	0.1598	0.2219	0.1851	1.0208	0.00000576	0.0031	0.0088	0.003	0.7102	-0.0024	0.0402	-0.0219	0.0149	-0.8208
	200	0.2006	0.1596	0.2069	0.1854	0.9273	0.00000036	0.003	0.0031	3.10E-03	0.5518	-0.0006	0.0404	-0.0069	0.0146	-0.7273
	300	0.1999	0.1558	0.2056	0.1918	0.8794	0.00000001	0.0033	0.0017	0.0042	0.4788	0.0001	0.0424	-0.0056	0.0082	-0.6794
	500	0.1997	0.1571	0.203	0.1932	0.8375	0.8375	0.0031	0.0009	0.0041	0.4192	0.0003	0.043	-0.003	0.0068	-0.6375

$$\begin{aligned} AIC =& 2logL + 2k, \\ CAIC =& -2logL + \frac{2kn}{(n-k-1)}, \\ BIC =& -2logL + klog(n), \\ HQIC =& -2logL + 2klog(log(n)) \end{aligned}$$

where L is the likelihood function, k is the number of parameters of the model and n is the sample size. By respecting the standards in the field, the best model corresponds to smaller $-logL, K\text{–}S, AIC, CAIC, BIC, HQIC$, and greater p-value. Here, we used the "AdequacyModel" package in R programming language to obtain the MLEs and goodness-of-fit tests of the given data sets.

The first data represents the total time on test plot analysis for mechanical components of the RSG-GAS reactor (Salman et al. (1999)) given below:

2.160 0.746 0.402 0.954 0.491 6.560 4.992 0.347 0.150 0.358 0.101 1.359 3.465 1.060 0.614 1.921 4.082 0.199 0.605 0.273 0.070 0.062 5.320.
The second dataset corresponds to the maintenance data reported on active repair times (hours) for an airborne communication transceiver (Von Alven (1964)) and its values are:

0.2 0.3 0.5 0.5 0.5 0.5 0.6 0.6 0.7 0.7 0.7 3 .8 1.0 1.0 1.0 1.0 1.1 1.3 1.5 1.5 1.5 1.5 2.0 2.0 2.2 2.5 2.7 3.0 3.0 3.3 3.3 4.0 4.0 4.5 4.7 5.0 5.4 5.4 7.0 7.5 8.8 9.0 10.3 22.0 24.5.

Table 4: Basic statistical description of the two datasets.

Data set	Size (n)	Min.	Max.	Mean	Median	SD	Skewness	Kurtosis
First	23	0.0620	6.5600	1.5780	0.6140	1.9307	1.3643	3.5446
Second	100	0.9200	5.3060	1.6580	1.5440	0.5994	3.1824	17.4236

Table 4 displays basic descriptive statistics of the two datasets. Here, the distribution of the dataset shows a positive skewness and leptokurtic behaviour, which goes with the moment properties of this distribution. Figure 3 shows the TTT plot of the two data sets and it goes with the features of hrf of YFD.

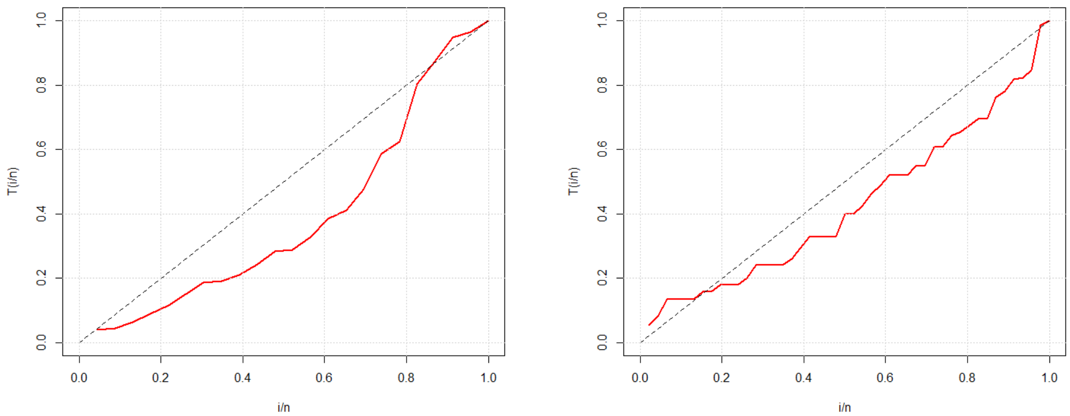


Figure 3: The TTT plots of the first data set (left) and the second data set(right).

Table 5: The MLEs of the first data set.

Model	**MLEs**	**-log L**
YFD	$\hat{\alpha} = 3.171614, \hat{\sigma} = 2.8807293, \hat{\lambda} = 0.3978812$	32.1474
YWD	$\hat{\alpha} = 1.1656874, \hat{\sigma} = 0.6226505, \hat{\lambda} = 0.8270732$	32.5615
TFD	$\hat{\alpha} = 0.7314193, \hat{\sigma} = 0.6346519, \hat{\lambda} = 0.8446795$	32.8500
EWD	$\hat{\alpha} = 1.4933965, \hat{\sigma} = 0.6235447, \hat{\lambda} = 1.7240681$	32.2272
YED	$\hat{\alpha} = 3.3114036, \hat{\sigma} = 0.2134588$	35.8682

Table 6: The goodness-of-fit statistics for the first data set.

Model	**YFD**	**YWD**	**TFD**	**EWD**	**YED**
W	0.0225	0.0665	0.0373	0.0491	0.0925
A	0.2200	0.4375	0.3266	0.3433	0.5850
AIC	70.2948	71.1229	71.7000	70.4544	75.7364
BIC	73.7013	74.5294	75.1065	73.8609	78.0074
CAIC	71.5580	72.3861	72.9632	71.7175	76.3364
HQIC	71.1515	71.9797	72.5567	71.3111	76.3075
K-S	0.0853	0.1306	0.0916	0.1156	0.1754
P-Value	0.9909	0.7808	0.9808	0.8959	0.4299

Table 5 shows the results of the MLEs and negative log-likelihood values. From Table 6 we can conclude that YFD provides the lowest W, A, AIC, BIC, CAIC, HQIC, K-S values, and the largest p-value. Therefore, YFD is chosen as the best fit for the first data. Figure 4 represents the fitted cdfs of the YFD, YWD, EWD, TFD, and YED models with the empirical distribution for the first data set.

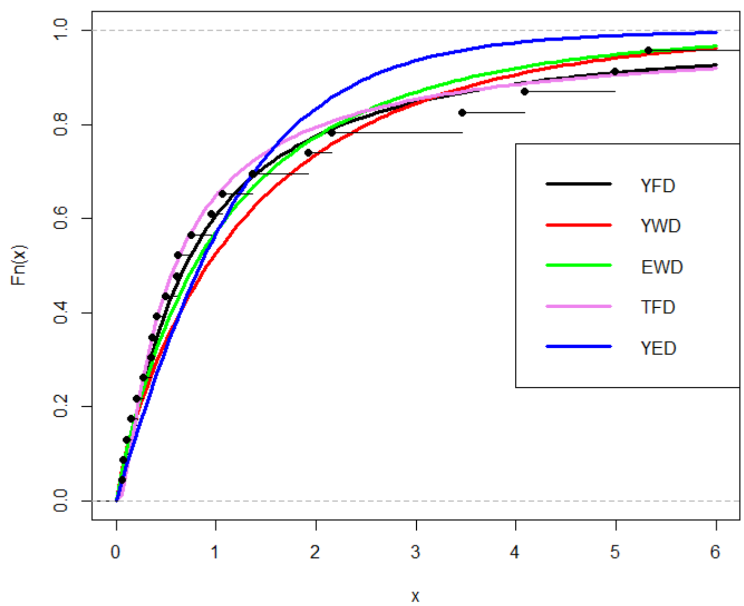


Figure 4: CDF plots for the first data set.

The performance of YFD for the second data was also compared to that of YWD, EWD, TFD, and YED. Table 7 lists MLEs of the competitive model parameters and $-logL$ values. The goodness-of-fit measures displayed in Table 8 prove that the YFD is the best fit for the second data set. Figure 5 represents the fitted cdfs of the YFD, YWD, EWD, TFD, and YED models with the empirical distribution for the second data set.

Table 7: The MLEs of the second data set.

Model	**MLEs**	**-log L**
YFD	$\hat{\alpha} = 2.0172188, \hat{\sigma} = 2.6584730, \hat{\lambda} = 0.6335838$	101.4790
YWD	$\hat{\alpha} = 2.4322088, \hat{\sigma} = 0.1218665, \hat{\lambda} = 0.9077587$	105.0426
TFD	$\hat{\alpha} = 1.9680747, \hat{\sigma} = 0.7954685, \hat{\lambda} = 0.8312829$	102.0101
EWD	$\hat{\alpha} = 0.5855134, \hat{\sigma} = 0.6815840, \hat{\lambda} = 1.9688407$	103.2205
YED	$\hat{\alpha} = 1.4732825, \hat{\sigma} = 0.1759881$	105.4382

Table 8: The goodness-of-fit statistics for the second data set.

Model	YFD	YWD	TFD	EWD	YED
W	0.0524	0.1072	0.0664	0.0720	0.1144
A	0.3155	0.7700	0.4070	0.5313	0.8312
AIC	208.9581	216.0851	210.0203	212.441	214.8764
BIC	214.444	221.571	215.5062	217.9269	218.5337
CAIC	209.5295	216.6565	210.5917	213.0124	215.1554
HQIC	211.0131	218.1402	212.0753	214.4961	216.2464
K-S	0.1015	0.1085	0.1099	0.1129	0.1435
P-Value	0.7302	0.6509	0.6348	0.6002	0.2996

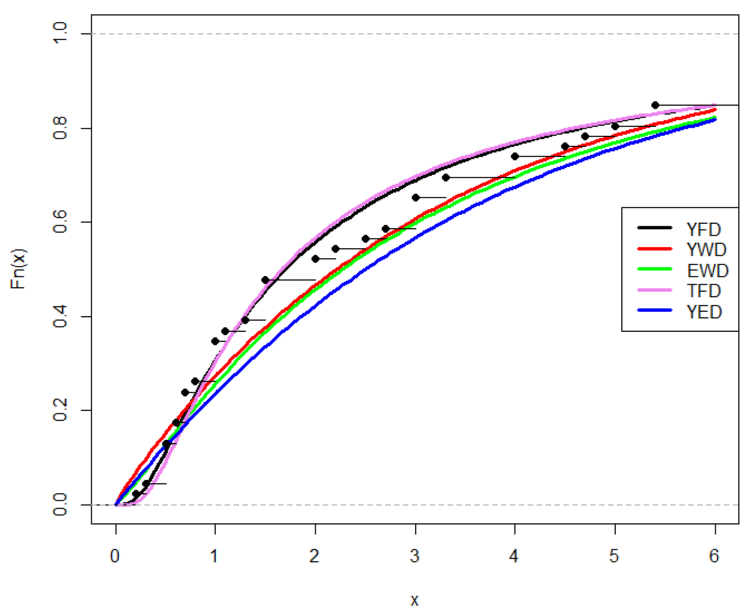


Figure 5: CDF plots for the second data set.

7 Conclusion

In this article, we proposed a versatile continuous distribution based on the Yun-G family, namely YFD. Several statistical properties of the proposed distribution, such as moments, skewness, kurtosis, stochastic ordering, and entropy are evaluated. Two characterizations of the distribution are obtained using the hazard rate function and truncated moments. Five methods of parametric estimation have been utilized in this study. The simulation study showed the accuracy and consistency of the estimation techniques. Two real-world data sets from the engineering sector were used to demonstrate the flexibility of the proposed model. In comparison to some competitive models, the YFD provided the best fit for the data sets.

Acknowledgements: The authors are grateful to the editor and reviewer for their valuable suggestions which helped in the improvement of the paper.

References

Abd El Khalek and Rania, H.M. (2022). The Generalized Odd Log-Logistic Fréchet Distribution for Modeling Extreme Values. *Pakistan Journal of Statistics and Operation Research*, 18, 649–674.

Akarawak, E., John, A., Khaleel, M., Adedayo, A. and Ogunsanya, A (2023). The inverted Gompertz-Fréchet distribution with applications. *Scientific African* 21, e01769.

Alzaatreh, A., Lee, C. and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71, 63-79.

Anderson, T.W. and Darling, D.A. (1952). Asymptotic Theory of Certain "Goodness of Fit" Criteria Based on Stochastic Processes. *The Annals of Mathematical Statistics*, 23, 193–212.

Aurise, N., Leo, O.O., Euna, N., Alexis, H. and Abdisalam, H.M (2023). Marshall-Olkin Exponentiated Fréchet Distribution. *Journal of Data Analysis and Information Processing*, 11, 262–292.

Casella, G. and Berger, R.L. (2002). *Statistical Inference*, Duxbury advanced series in statistics and decision sciences, Thomson Learning.

Chesneau, C., Babu, M.G. and Bakouch, H.S. (2021). The Yun transform in probabilistic and statistical contexts: Weibull baseline case and its applications in reliability theory. *Journal of Computational Mathematics and Data Science*, 1, 100002.

Galambos, J. and Kotz, S. (1978). Characterizations of probability distributions. A unified approach with an emphasis on exponential and related models. *Lecture Notes in Mathematics, 675. Berlin, Germany: Springer.*

Glänzel, W. (1987). A characterization based on truncated moments and its application to some distribution families. *Mathematical Statistics and Probability Theory*, B, 75–84.

Glänzel, W., Telcs, A. and Schubert, A. (1984). Characterization by truncated moments and its application to Pearson-type distributions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 66, 173–183.

Glänzel, W. (1988). A characterization of the normal distribution. *Studia Scientiarum Mathematicarum Hungarica*, 2, 89–91.

Glänzel, W. (1990). Some consequences of a characterization theorem based on truncated moments. *Statistics*, 21, 613–618.

Hamedani, G.G. (1993). Characterizations of Cauchy, normal, and uniform distributions. *Studia Scientiarum Mathematicarum Hungarica*, 3, 243–248.

Hamedani, G.G. and Ahsanullah, M. (2005). Characterizations of univariate continuous distributions based on hazard function II. *Journal of Statistical Theory and Applications*, 4, 218–238.

James, J.S., Sekhar, V. and James, R.W. (1988). Least-squares estimation of distribution functions in Johnson's translation system. *Journal of Statistical Computation and Simulation*, 29, 271–297.

Jayakumar, K. and Babu, M.G. (2019). A New Generalization of the Fréchet Distribution: Properties and Application. *Statistica*, 79, 267–289.

Kotz, S. and Shanbhag, D.N. (1980). Some new approaches to probability distributions. *Advances in Applied Probability*, 12, 903–921.

Macdonald, P.D.M. (2018). Comments and Queries Comment on "An Estimation Procedure for Mixtures of Distributions" by Choi and Bulgren. *Journal of the Royal Statistical Society: Series B (Methodological)*, 33, 326–329.

Mahmoud, M.R. and Mandouh, R.M. (2013). On the Transmuted Fréchet Distribution. *Journal of Applied Sciences Research*, 9, 5553–5561.

Marshall, A.W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84, 641–652.

Najwan, A., Aijaz, A., Muzamil, J., Ahmed, M.G., Mohammed, A.M., Eslam, H., Ehab, M.E. and Moyazzem, M.H. (2023). The novel Kumaraswamy power Fréchet distribution with data analysis related to diverse scientific areas. *Alexandria Engineering Journal*, 70, 651–664.

Pal, M., Ali, M.M. and Woo, J. (2006). Exponentiated Weibull distribution. *Statistica*, 66, 139–147.

Rényi, A. (1961). On measures of entropy and information. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, 1: Contributions to the Theory of Statistics.* 4, 547–562.

Salman, S.M., Suprawhardana and Prayoto, S. (1999). Total time on test plot analysis for mechanical components of the RSG-GAS reactor. *Atom Indones*, 25, 155–61.

Shaked, M. and Shanthikumar, J.G. (2007). *Stochastic orders*, Springer.

Shalabi, R. (2020). Cubic Transmuted Fréchet Distribution. *Journal of Applied Sciences Research*, 16, 9–19.

Shannon, C.E. (1948). A Mathematical Theory of Communication. *Bell System Technical Journal*, 27, 379–423.

Von Alven, W.H. (1964), *Reliability Engineering.* Prentice-Hall, Englewood Cliffs, New Jersey.

Yaming, Y.U. (2009). Stochastic Ordering of Exponential Family Distributions and Their Mixtures. *Journal of Applied Probability*, 46, 244–254.

Yun, B.I. (2014). An ad hoc approximation to the Gauss error function and a correction method. *Applied Mathematical Sciences*, 8, 4261–4273.