

GENERALIZED T – X FAMILY OF DISTRIBUTIONS AND APPLICATIONS

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ABSTRACT

In this paper, a new method of generating families of distributions using any p.d.f. as a generator which is called Transformed – Transformer (T – X) family of distributions is considered. Some members of this family including Gumbel – Uniform, Gumbel – Pareto, Gumbel – Frechet and their mathematical properties are explored. The method of maximum likelihood is used for estimating the distribution parameters. The flexibility of the new model is illustrated using real life data sets. Also industrial reliability test plans for acceptance or rejection of a lot of products submitted for inspection are developed. The acceptance sampling plan proposed here can save the test time in practical situations. Marshall – Olkin generalization of T – X family is also introduced. Stress-strength analysis is discussed. A simulation study is also conducted to study the behaviour of estimate of stress-strength reliability. Auto-regressive models are discussed based on Marshall – Olkin Gumbel uniform distribution and sample path properties are explored. Transmutation technique is also considered in T – X family. A real data is used to explain the flexibility of Gumbel Transmuted Uniform model, a member of Gumbel Transmuted X family.

Key words and Phrases: *Auto-regressive process; Minification processes; max - min processes; Operating Characteristic Function; Reliability test plan; Stress - strength analysis; Transmuted family. .*

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1 Introduction

People use probability models for making wise decisions in the face of uncertainty. Probability distributions are vital and fundamental to statistical theory and are concerned with creating the basis and foundation for the study of uncertainty. The interest in developing more flexible statistical distributions remains strong in statistics and related areas. Recently, many generalized classes of distributions have been developed and applied to model data sets from various contexts. $T - X$ family of distributions is one among them. Alzaatreh et al. (2013) proposed a general method of generating new family of distributions with any continuous random variable T as the generator and another function $U(F(x))$ which satisfies the following conditions:

- $U(F(x)) \in [a, b]$
- $U(F(x))$ is differentiable and monotonically non - decreasing.
- $U(F(x)) \longrightarrow a$ as $x \longrightarrow -\infty$ and $U(F(x)) \longrightarrow b$ as $x \longrightarrow \infty$

The new class of distributions is defined by

$$G(x) = \int_a^{U(F(x))} r(t)dt = R[U(F(x))] \quad (1.1)$$

where $R(t)$ is the cdf and $r(t)$ is the pdf of the random variable T . Here, the cdf in (1.1) is a composite function of $(R, U, F)(x)$. The corresponding pdf is

$$g(x) = \left\{ \frac{d}{dx} U(F(x)) \right\} r\{U(F(x))\} \quad (1.2)$$

The p.d.f. $r(t)$ in (1.2) is “transformed” into a new pdf $g(x)$ through the function $U(F(x))$, which acts as a “transformer”. That is, a random variable X , “the transformer”, is used to transform another random variable T , “the transformed”. The resulting family is known as “Transformed - Transformer” or “ $T - X$ ” family of distributions. A large number of distributions, continuous and discrete, can be generated by applying any two existing univariate distributions based on this method. Alzaatreh et al. (2013 b) gave several choices of $U(F(x))$ depending upon the support of the random variable T . Some examples are

- When the support of T is bounded or support of T is $[0,1]$: In this case $U(F(x))$ can be taken as $F(x)$ or $F^\alpha(x)$. This leads to the beta - generated family of distributions.
- When the support of T is $[0, \infty)$, $a \geq 0$: Without loss of generality, we assume $a = 0$. Then $U(F(x))$ can be defined as $-\log(1 - F(x))$, $F(x)/(1 - F(x))$, $-\log(1 - F^\alpha(x))$ and $F^\alpha(x)/(1 - F^\alpha(x))$, where $\alpha > 0$.
- When the support of T is $(-\infty, \infty)$: Then $U(F(x))$ can be taken as $\log[-\log(1 - F(x))]$, $\log[F(x)/(1 - F(x))]$, $\log[-\log(1 - F^\alpha(x))]$ and $\log[F^\alpha(x)/(1 - F^\alpha(x))]$.

By taking $U(F(x))$ as $-\log(1 - F(x))$, Alzaatreh et al. (2012 a, b, 2013) studied Gamma - Pareto distribution, T - geometric family, a discrete analogue of the distribution of any continuous random variable T and Weibull - Pareto distribution. By including an additional parameter, Alzaghal (2013) introduced exponentiated $T - X$ family of distributions which provide more flexibility for fitting real data. Alzaatreh et al. (2014 a, b) introduced T - normal family of distributions as a new approach to generalize the normal distribution and gamma - normal distribution as a special case. Aljarrah et al. (2014) proposed the use of quantile function of another random variable to define the U function. One advantage of using this approach is that one can keep one or more parameters from the distribution used for finding quantile function. Alzaatreh and Ghosh (2015) proposed the Weibull - X family of distributions. In particular, keeping a shape parameter from this distribution can add more flexibility to the new distribution. Recently, the Logistic - X family of distributions was introduced by Tahir et al. (2016). This paper is organized as follows. In Section 1 the T-X family of distributions and its general theory are discussed. In Section 2, we concentrate on the Gumbel - X family of distributions which resulted in developing a number of new distributions namely Gumbel - Uniform, Marshall - Olkin generalized Gumbel - Uniform, Gumbel - Pareto and Gumbel - Fréchet distributions. In Section 3, we consider stress- strength analysis. Section 4 deals with Industrial reliability test plan and acceptance sampling numbers. In Section 5 we introduce Marshall Olkin Extended Gumbel - X Family of distributions and study their properties. Section 6 deals with Auto-regressive time series modeling, minification processes and max-min auto-regressive processes. In section 7, we estimate the reliability and obtain confidence intervals using a simulation study. In section 8, we developed transmuted Gumbel - Pareto and Gumbel transmuted X family of distributions. In section 9, a real data set is used to test the flexibility of newly developed Gumbel Transmuted Uniform distribution. The conclusions are given in section 10.

The Gumbel distribution is one of the most commonly used models to carry out risk

analysis in extreme events such as flood levels and wave heights, in reliability tests and in life expectancy experiences. The potential applicability of the Gumbel distribution to represent the distribution of maxima relates to extreme value theory which indicates that it is likely to be useful if the distribution of the underlying sample data is of the normal or exponential type.

Al - Aqtash (2013) and Al - Aqtash et al. (2014, 2015) proposed the Gumbel - X family by taking T as the Gumbel random variable with cdf is given by

$$R(t) = e^{-e^{-\frac{t-\mu}{\sigma}}}, \quad -\infty < t < \infty \quad (1.3)$$

The pdf corresponding to (1.3) is given by

$$r(t) = \frac{1}{\sigma} e^{-\frac{t-\mu}{\sigma}} e^{-e^{-\frac{t-\mu}{\sigma}}}, \quad -\infty < t < \infty \quad (1.4)$$

Let $F(x)$ and $\bar{F}(x) = 1 - F(x)$ be the baseline cdf and survival function (sf) depending on a parameter vector Θ . By replacing upper limit $U[F(x)]$ with $\log \left[\frac{F(x)}{\bar{F}(x)} \right]$ and $r(t)$ with Gumbel density given in (1.4) in (1.1), we define the cdf of the Gumbel - X family by

$$G(x) = e^{-e^{\frac{\mu}{\sigma}} \left(\frac{F(x)}{\bar{F}(x)} \right)^{-1/\sigma}}. \quad (1.5)$$

The pdf of the family reduces to

$$g(x) = \frac{e^{\mu/\sigma}}{\sigma} f(x) \frac{(F(x))^{-\frac{1}{\sigma}-1}}{(\bar{F}(x))^{-\frac{1}{\sigma}+1}} e^{-e^{\frac{\mu}{\sigma}} \left(\frac{F(x)}{\bar{F}(x)} \right)^{-1/\sigma}}. \quad (1.6)$$

The support of the random variable associated with $g(x)$ and $f(x)$ are equal.

Lemma 1.1. *Let X have a density of the form (1.6). The random variable $T = \log \left[\frac{F(X)}{\bar{F}(X)} \right]$ has a Gumbel distribution denoted by $Gu(\mu, \sigma)$ distribution. The converse is also true, that is, if T has a Gumbel distribution i.e., $Gu(\mu, \sigma)$, then the random variable $X = F^{-1}(e^T(1 + e^T)^{-1})$ has a Gumbel - X distribution with density (1.6).*

2 Members of Gumbel – X family

Here we introduce some members of Gumbel -X family.

2.1 Gumbel - Uniform Distribution

If the parent distribution is uniform in the interval (a, b) , then the pdf, cdf, and sf are given by $f(x; a, b) = \frac{1}{b-a}$, $a < x < b$, $F(x; a, b) = \frac{x-a}{b-a}$ and $\bar{F}(x) = \frac{b-x}{b-a}$.

Then the Gumbel - Uniform (GuU) cdf is given by

$$G_{GuU}(x; \mu, \sigma, a, b) = e^{-e^{\mu/\sigma} \left(\frac{x-a}{b-x} \right)^{-\frac{1}{\sigma}}}, \quad a < x < b. \quad (2.1)$$

and the pdf is given by

$$g_{GuU}(x; \mu, \sigma, a, b) = \frac{(b-a)e^{\mu/\sigma}}{\sigma} \frac{(x-a)^{-\frac{1}{\sigma}-1}}{(b-x)^{-\frac{1}{\sigma}+1}} e^{-e^{\mu/\sigma} \left(\frac{x-a}{b-x} \right)^{-\frac{1}{\sigma}}}, \quad a < x < b. \quad (2.2)$$

Here, the range of Gumbel - Uniform distribution depends on the parameters a and b . So, this distribution can be used for modeling and analyzing any data set by changing these parameters.

2.2 Gumbel - Pareto Distribution

If the parent distribution is Pareto with parameters k and θ , then its cdf is

$$F(x) = 1 - \left(\frac{x}{\theta} \right)^{-k}, \quad x > \theta \quad (2.3)$$

The pdf and sf corresponding to (2.3) are

$$f(x) = \frac{k}{\theta} \left(\frac{x}{\theta} \right)^{-k-1} \quad (2.4)$$

and

$$\bar{F}(x) = \left(\frac{x}{\theta} \right)^{-k}. \quad (2.5)$$

2.3 Gumbel - Fréchet Distribution

If X follows Fréchet distribution with parameters α and β ,

$$G(x) = \exp \left\{ -\lambda \left[\frac{e^{-(\frac{\alpha}{x})^\beta}}{1 - e^{-(\frac{\alpha}{x})^\beta}} \right]^{-1/\sigma} \right\}. \quad (2.6)$$

The pdf is

$$g(x) = \frac{\lambda\beta}{\alpha\sigma} \left(\frac{\alpha}{x} \right)^{\beta+1} e^{(\alpha/x)^\beta} \left[e^{(\alpha/x)^\beta} - 1 \right]^{1/\sigma-1} \exp \left\{ -\lambda \left[\frac{e^{-(\frac{\alpha}{x})^\beta}}{1 - e^{-(\frac{\alpha}{x})^\beta}} \right]^{-1/\sigma} \right\}. \quad (2.7)$$

3 Stress - Strength Analysis

The reliability is defined as the probability of not failing, denoted by R and is defined as $R = P(X < Y)$ where X represents the stress and Y represents the strength of a component. For the evaluation of R , here we assume that both the random variables follow the distributions belonging to the same family and are independent. There are a number of applications in the literature including stress - strength model and breakdown of a system having two components. Other applications of the reliability parameter can be found in Weerahandi and Johnson (1992). If X and Y are two independent random variables with cdf $F_1(x)$ and $F_2(y)$ and pdf $f_1(x)$ and $f_2(y)$ respectively. Then

$$R = P(X < Y) = \int_{-\infty}^{\infty} F_2(t) f_1(t) dt. \quad (3.1)$$

Lemma 3.1. *If X and Y are two independent random variables following Gumbel - X family of distributions with parameters (λ_1, σ_1) and (λ_2, σ_2) respectively. Then*

$$R = \sum_{j=0}^{\infty} \frac{(-1)^j \lambda_2^j}{j! \lambda_1^{\frac{j\sigma_1}{\sigma_2}}} \Gamma(j \frac{\sigma_1}{\sigma_2} + 1) \quad (3.2)$$

Proof. Substituting the cdf (1.5) and pdf (1.6) of Gumbel - X distribution after replacing $e^{\mu/\sigma} = \lambda$ in (3.2) and $\log[\frac{F(x)}{F(x)}] = u$ we get,

$$R = \frac{\lambda_1}{\sigma_1} \int_{-\infty}^{\infty} e^{-u/\sigma_1} e^{-\lambda_2 e^{-u/\sigma_2}} e^{-\lambda_1 e^{-u/\sigma_1}} du$$

By substituting $t = \lambda_1 e^{-u/\sigma_1}$ and solving we get,

$$R = \sum_{j=0}^{\infty} \frac{(-1)^j \lambda_2^j}{j! \lambda_1^{\frac{j\sigma_1}{\sigma_2}}} \int_0^{\infty} e^{-t} t^{j \frac{\sigma_1}{\sigma_2}} dt$$

Using $\int_0^{\infty} e^{-t} t^{j \frac{\sigma_1}{\sigma_2}} dt = \Gamma(j \frac{\sigma_1}{\sigma_2} + 1)$, the gamma function, we get the result. $\square$

Reliability measure (R) for various values of λ_1 when $\sigma_1 = \sigma_2 = 1$ and $\lambda_2 = 1$ are listed in Table 4.4.

Using the values of R , we can device system which ensure reliability values as given above and development of such system is an engineering problem.

Table 1: Reliability measure (R) for various values of λ_1

λ_1	5	10	15	20	25	30	35	40	50	60
R	0.83	0.909	0.938	0.952	0.962	0.968	0.972	0.975	0.980	0.98

4 Industrial Reliability Test Plan

In industrial quality control, we consider the reliability test plan for accepting or rejecting a lot by assuming that the life time of the products follow Gumbel - X distribution. A common procedure in life testing is to terminate the test by a pre - determined 't' and note the number of failures (assuming that a failure is well defined). The decision to accept the specific average life occurs if and only if the number of observed failures at the end of the fixed time 't' does not exceed 'c', called the acceptance number. When the number of failures exceeds 'c', the test may be terminated before the time 't' and reject the lot. For such a truncated life test and the associated decision rule, we are interested in obtaining the smallest sample size to arrive at a decision. Consider the special case of Gumbel - Uniform distribution by substituting $a = 0$ and $b = \theta$ with

$$G_{GuU}(x; \mu, \sigma, 0, \theta) = e^{-e^{\mu/\sigma}(\frac{\theta}{x}-1)^{\frac{1}{\sigma}}}. \quad (4.1)$$

Here the average life time depends only on θ if μ and σ are known.

Let θ_0 be the required minimum average life time. Then

$$G_{GuU}(x; \mu, \sigma, 0, \theta) \leq G_{GuU}(x; \mu, \sigma, 0, \theta_0) \Leftrightarrow \theta \geq \theta_0. \quad (4.2)$$

A sampling plan is specified by the following quantities:

- the number of units n on test,
- the acceptance number c ,
- the maximum test duration t and
- the minimum average lifetime represented by θ_0 .

The consumer's risk, i.e. the probability of accepting a bad lot should not exceed the value $1 - p^*$, where p^* is a lower bound for the probability that a lot of true value θ below θ_0 is rejected by the sampling plan. For fixed p^* the sampling plan is characterized by $(n, c, t/\theta_0)$. By considering sufficiently large lots, binomial distribution can be used to find the acceptance probability. The problem is to determine the smallest positive integer n for given values of c and t / θ_0 such that

$$L(p_0) = \sum_{i=0}^c \binom{n}{i} p_0^i (1 - p_0)^{n-i} \leq 1 - p^* \quad (4.3)$$

where $p_0 = G(t; \mu, \sigma, 0, \theta_0)$ given by (4.1) which indicates failure probability before time ' t ' which depends only on the ratio t/θ_0 . The function $L(p)$ is called the operating characteristic function of the sampling plan, i.e. the acceptance probability of the lot as a function of the failure probability $p(\theta) = G(t; \mu, \sigma, 0, \theta)$.

The average life time of the product is increasing with θ and the failure probability $p(\theta)$ decreases implying that the operating characteristic function is increasing in θ . The minimum values of n satisfying (4.3) are obtained for $\mu = 1, \sigma = 1$ and $p^* = 0.75, 0.90, 0.95, 0.99$ and $t / \theta_0 = 0.6, 0.65, 0.7, 0.75, 0.8, 0.85, 0.9$ and 0.95 . The results are displayed in Table 2.5.

If $p_0 = G(t; \mu, \sigma, 0, \theta_0)$ is small and n is very large, the binomial probability may be approximated by Poisson probability with parameter $\lambda = np_0$ so that (4.3) becomes

$$L_1(p_0) = \sum_{i=0}^c \frac{\lambda^i}{i!} e^{-\lambda} \leq 1 - p^*. \quad (4.4)$$

The minimum values of n satisfying (4.4) are obtained for the same combination of values of μ, σ, p^* and t/θ_0 and are presented in Table 2.6.

The operating characteristic function of the sampling plan $(n, c, t/\theta_0)$ gives the probability $L(p)$ of accepting the lot with

$$L(p) = \sum_{i=0}^c \binom{n}{i} p_0^i (1 - p_0)^{n-i} \quad (4.5)$$

where $p = G(t, \theta)$ is considered as a function of θ .

Table 2: Minimum sample size using binomial approximation

p^*	c	t/θ_0							
		0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95
0.75	0	8	6	4	3	2	2	2	1
	1	16	11	8	6	5	4	3	2
	2	24	16	12	9	7	6	5	4
	3	31	21	16	12	9	8	6	5
	4	38	26	19	15	12	9	8	6
	5	45	31	23	18	14	11	9	7
	6	52	36	27	20	16	13	10	9
	7	58	41	30	23	18	15	12	10
	8	65	46	34	26	20	16	13	11
	9	72	50	37	28	22	18	15	12
	10	79	55	41	31	24	20	16	13
0.90	0	13	9	7	5	4	3	2	2
	1	23	16	11	9	7	5	4	3
	2	31	22	16	12	9	7	6	4
	3	40	27	20	15	12	9	7	6
	4	47	33	24	18	14	11	9	7
	5	55	38	28	21	16	13	10	8
	6	63	44	32	24	19	15	12	10
	7	70	49	36	27	21	17	13	11
	8	77	54	39	30	23	19	15	12
	9	85	59	43	33	26	20	16	13
	10	92	64	47	36	28	22	18	15

The values of n and c are determined by means of operating characteristics (OC) function for given value of p^* and t / θ_0 are displayed in Table 2.7 by considering the fact that $p = G(\frac{t}{\theta_0} | \frac{\theta}{\theta_0})$.

The producer's risk is the probability of rejecting a lot when $\theta > \theta_0$. For the given value of producer's risk say 0.05, we obtain p from the sampling plan given in Table 2.5 subject to the condition that

$$\sum_{i=0}^c \binom{n}{i} p_0^i (1 - p_0)^{n-i} \geq 0.95 \quad (4.6)$$

The minimum value of θ/θ_0 satisfying (4.6) for the sampling plan $(n, c, t / \theta_0)$ and for the given p^* are listed in Table 2.8.

Table 3: Minimum sample size using Poisson approximation

p^*	c	t/θ_0							
		0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95
0.75	0	9	6	4	4	3	3	2	2
	1	17	12	9	7	6	5	4	4
	2	25	17	13	10	8	7	6	5
	3	32	23	17	13	11	9	7	6
	4	39	28	21	16	13	11	9	8
	5	46	33	24	19	15	12	11	9
	6	53	37	28	22	17	14	12	10
	7	60	42	32	24	20	16	14	12
	8	67	47	35	27	22	18	15	13
	9	74	52	39	30	24	20	17	14
	10	80	57	42	33	26	22	18	16
0.90	0	15	10	8	6	5	4	4	3
	1	24	17	13	10	8	7	6	5
	2	33	24	18	14	11	9	8	7
	3	41	29	22	17	14	11	10	8
	4	49	35	26	20	16	13	11	10
	5	57	41	30	23	19	15	13	11
	6	65	46	34	27	21	18	15	13
	7	73	51	38	30	24	20	16	14
	8	80	57	42	33	26	21	18	15
	9	87	62	46	36	29	23	20	17
	10	95	67	50	39	31	25	21	18

Table 4: Values of the Operating Characteristic function for the sampling plan $(n, c, t/\theta_0)$

				θ/θ_0					
p^*	n	c	t/θ_0	1	1.25	1.5	1.75	2	2.25
0.75	24	2	0.6	0.2245	0.8702	0.9924	0.9997	0.9999	0.9999
	16	2	0.65	0.2476	0.8639	0.9901	0.9994	0.9999	0.9999
	12	2	0.7	0.2253	0.8388	0.9855	0.9990	0.9999	0.9999
	9	2	0.75	0.2242	0.8296	0.9822	0.9986	0.9999	0.9999
	7	2	0.8	0.2155	0.8219	0.9791	0.9981	0.9998	0.9999
	6	2	0.85	0.1540	0.7835	0.9708	0.9969	0.9997	0.9999
	5	2	0.9	0.1151	0.7687	0.9664	0.9959	0.9996	0.9999
	4	2	0.95	0.0886	0.7923	0.9699	0.9962	0.9995	0.9999
0.90	31	2	0.6	0.0985	0.7780	0.9846	0.9993	0.9999	0.9999
	22	2	0.65	0.08737	0.7361	0.9760	0.9986	0.9999	0.9999
	16	2	0.7	0.0831	0.7086	0.9676	0.9977	0.9999	0.9999
	12	2	0.75	0.0792	0.6894	0.9597	0.9965	0.9997	0.9999
	9	2	0.8	0.0833	0.6931	0.9563	0.9956	0.9996	0.9999
	7	2	0.85	0.0790	0.6958	0.9536	0.9948	0.9995	0.9999
	6	2	0.9	0.0435	0.6522	0.9409	0.9924	0.9991	0.9999
	4	2	0.95	0.0886	0.79223	0.9699	0.9962	0.9995	0.9999

Table 5: Minimum ratio of true θ and required θ_0 for the acceptability of a lot with producer's risk of 0.05 for $\mu = \sigma = 1$

p^*	c	t/θ_0							
		0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95
0.75	0	1.0904	1.1514	1.1946	1.2455	1.3208	1.3956	1.4300	1.4500
	1	0.9750	1.0142	1.0600	1.1090	1.1596	1.1930	1.2174	1.2189
	2	0.9300	0.9611	0.9974	1.0385	1.0710	1.1076	1.1466	1.1731
	3	0.9016	0.9339	0.9757	1.0145	1.0342	1.0829	1.0993	1.1229
	4	0.8872	0.9190	0.9575	0.9945	1.0265	1.0438	1.0830	1.0891
	5	0.8746	0.9060	0.9469	0.9773	1.0057	1.0329	1.0594	1.0679
	6	0.8666	0.9001	0.9371	0.9623	0.9935	1.0229	1.0391	1.0746
	7	0.8577	0.8912	0.9240	0.9555	0.9878	1.0181	1.0391	1.0614
	8	0.8522	0.8891	0.9199	0.9490	0.9772	1.0026	1.0247	1.0493
	9	0.8483	0.8792	0.9122	0.9429	0.9722	1.0006	1.0282	1.0407
	10	0.8445	0.8774	0.9092	0.9399	0.9650	0.9985	1.0180	1.0327
0.90	0	1.1569	1.2002	1.2722	1.3286	1.3886	1.4203	1.4625	1.5438
	1	1.0026	1.0563	1.0922	1.1517	1.1964	1.2321	1.2632	1.2850
	2	0.9536	0.9981	1.0350	1.0686	1.1078	1.1379	1.1728	1.1731
	3	0.9300	0.9653	1.0023	1.0385	1.0821	1.0989	1.1244	1.1604
	4	0.9086	0.9422	0.9825	1.0145	1.0513	1.0755	1.1052	1.1229
	5	0.8931	0.9277	0.9693	0.9992	1.0265	1.0619	1.0780	1.0929
	6	0.8834	0.9262	0.9575	0.9856	1.0192	1.0496	1.0731	1.0969
	7	0.8763	0.9123	0.9469	0.9773	1.0057	1.0383	1.0551	1.0817
	8	0.8697	0.9036	0.9371	0.9696	0.9995	1.0329	1.0510	1.0679
	9	0.8635	0.8989	0.9299	0.9623	0.9935	1.0158	1.0391	1.0583
	10	0.8606	0.8923	0.9265	0.9589	0.9878	1.0113	1.0391	1.0614

Table 6: Operating characteristic values for $n = 9$, $c = 2$ and $t/\theta_0 = 0.75$

θ/θ_0	1	1.25	1.5	1.75	2	2.25
L(p)	0.2242	0.8296	0.9822	0.9986	0.9999	0.9999

5 Illustration of the Tables

Assume that the lifetime distribution is GuU distribution with $\mu = 1$ and $\sigma = 1$. Suppose that the experimenter is interested in establishing that the true unknown average life is at least 1000 hours with confidence $p^* = 0.75$. It is desired to stop the experiment at $t = 750$ hours. Then, for an acceptance number $c = 2$, the required n in Table 2.5 is 9. If during 750 hours, no more than 2 failures out of 9 are observed, then the experimenter can assert, with a confidence level of 0.75 that the average life is at least 1000 hours. If the Poisson approximation to binomial probability is used, the value of n from Table 2.6 is 10. For this sampling plan ($n = 9, c = 2, t/\theta_0 = 0.75$) under the GuU distribution the operating characteristic values from Table 2.7 are given in Table 2.9.

Comparing with Reliability Test Plans for Marshall - Olkin Gumbel - Maximum distribution (see Rani, 2011), for $\alpha = 3$, acceptance number $c = 10$, for the specified ratio $t/\theta_0 = 0.75$ and confidence level $p^* = 0.75$, the minimum sample size is 41 using binomial approximation and 44 using Poisson approximation, whereas for Gumbel - Uniform distribution it is 31 and 33 respectively. This improvement makes the new test plan more advantageous and helps in making optimal decisions.

The values in Table 2.9 show that if the true mean lifetime is twice the required mean lifetime ($\theta/\theta_0 = 2$), the producer's risk is approximately negligible.

6 Application to software reliability testing

Consider the following ordered failure times of the release of a software given in terms of hours from starting of the execution of the software up to the time at which a failure of the software is occurred (Wood, 1996). This data can be regarded as an ordered sample of size $n = 9$ consisting of the observations:

$$\{254, 788, 1054, 1393, 2216, 2880, 3593, 4281, 5180\}$$

Let the required average lifetime be 1000 hours and the testing time be $t = 750$ hours, this leads to ratio of $t/\theta_0 = 0.750$ with a corresponding sample size $n = 9$ and an acceptance number $c = 2$, which are obtained from Table 2.5 for $p^* = 0.75$. Therefore, the sampling plan for the above sample data is $(n = 9, c = 2, t/\theta_0 = 0.750)$. Based on the observations, we have to decide whether to accept the product or reject it. We accept the product only if the number of failures before 750 hours is less than or equal to 2. However, the confidence level is assured by the sampling plan only if the given life times follow Gumbel - Uniform distribution. In order to confirm that the given sample is generated by lifetimes following the Gumbel - Uniform distribution, we have compared the sample quantiles and the corresponding population quantiles and found a satisfactory agreement. Thus, the adoption of the decision rule of the sampling plan seems to be justified. In the sample of 9 units only, there is only one failure at 254 hours before $t = 750$ hours. Therefore we accept the product.

7 Marshall - Olkin Extended Gumbel - X family of Distributions

A new generalized family of distributions $\{S(x)\}$ called Marshall Olkin family was introduced by Marshall and Olkin (1997) by adding one more parameter to an existing family $\{G(x)\}$. Then the survival function is defined as

$$\bar{S}(x; \alpha) = \frac{\alpha \bar{G}(x)}{1 - \bar{\alpha} \bar{G}(x)}, \quad -\infty < x < \infty, \quad \alpha > 0 \quad (7.1)$$

Then the survival function of Marshall - Olkin Extended Gumbel -X family is obtained by substituting the survival function of Gumbel -X family

$$\bar{G}(x) = 1 - e^{-\lambda \left[\frac{F(x)}{\bar{F}(x)} \right]^{-1/\sigma}} \quad (7.2)$$

in (7.1) and is given by

$$\bar{S}(x, \alpha) = \frac{\alpha \{1 - e^{-\lambda \left(\frac{F(x)}{\bar{F}(x)} \right)^{-1/\sigma}}\}}{1 - \bar{\alpha} \{1 - e^{-\lambda \left(\frac{F(x)}{\bar{F}(x)} \right)^{-1/\sigma}}\}}. \quad (7.3)$$

7.1 Marshall - Olkin Extended Gumbel - Uniform Distribution

Suppose $X \sim U(0, \theta)$, where $\theta > 0$. Then $\bar{F}(x) = 1 - x/\theta$ and $F(x) = x/\theta$. Substituting in (7.3) we get the survival function as,

$$\bar{S}(x, \alpha) = \frac{\alpha\{1 - e^{-\lambda(\frac{\theta}{x}-1)^{1/\sigma}}\}}{1 - \bar{\alpha}\{1 - e^{-\lambda(\frac{\theta}{x}-1)^{1/\sigma}}\}}, \quad 0 < x < \theta, \alpha > 0. \quad (7.4)$$

and the cumulative distribution function is

$$K(x) = \frac{e^{-\lambda(\frac{\theta}{x}-1)^{1/\sigma}}}{1 - \bar{\alpha}\{1 - e^{-\lambda(\frac{\theta}{x}-1)^{1/\sigma}}\}}. \quad (7.5)$$

Then the pdf is

$$\begin{aligned} k(x) &= \frac{\alpha g(x)}{[1 - \bar{\alpha}\bar{G}(x)]^2} \\ &= \frac{\lambda\alpha\theta[\frac{\theta}{x} - 1]^{\frac{1}{\sigma}-1} \exp[-\lambda(\frac{\theta}{x} - 1)^{1/\sigma}]}{\sigma x^2 \{1 - \bar{\alpha}\{1 - \exp[-\lambda(\frac{\theta}{x} - 1)^{1/\sigma}]\}\}^2}. \end{aligned} \quad (7.6)$$

The hazard function is given by

$$h(x) = \frac{\lambda\theta(\frac{\theta}{x} - 1)^{1/\sigma-1} \exp[-\lambda(\frac{\theta}{x} - 1)^{1/\sigma}]}{\sigma x^2 \{1 - \bar{\alpha}\{1 - \exp[-\lambda(\frac{\theta}{x} - 1)^{1/\sigma}]\}\} \{1 - \exp[-\lambda(\frac{\theta}{x} - 1)^{1/\sigma}]\}}. \quad (7.7)$$

Theorem 7.1. Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d random variables with common survival function $\bar{G}(x)$. Let N be a geometric random variable independently distributed of $\{X_i, i \geq 1\}$ such that $P(N = n) = p(1 - p)^{n-1}$, $n = 1, 2, \dots$, $0 < p < 1$. Let $U_N = \min_{1 \leq i \leq N} X_i$. Then $\{U_N\}$ is distributed as $MOGuU(p, \lambda, \sigma, \theta)$ if and only if $X_i \sim GuU(\lambda, \sigma, \theta)$.

Proof. The survival function of the random variable $\{U_N\}$ is

$$\begin{aligned}
 \bar{H}(x) &= P(U_N > x) \\
 &= \sum_{n=1}^{\infty} P(U_n > x) P(N = n) \\
 &= \sum_{n=1}^{\infty} [\bar{G}(x)]^n p(1-p)^{n-1} \\
 &= \frac{p\bar{G}(x)}{1 - (1-p)\bar{G}(x)} \\
 &= \frac{p\{1 - e^{-\lambda(\frac{\theta}{x}-1)^{1/\sigma}}\}}{1 - \bar{p}\{1 - e^{-\lambda(\frac{\theta}{x}-1)^{1/\sigma}}\}}
 \end{aligned}$$

which is the survival function of MOGuU $(p, \lambda, \sigma, \theta)$. The converse easily follows. $\square$

Theorem 7.2. *Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d random variables with common survival function $\bar{G}(x)$. Let N be a geometric random variable independently distributed of $\{X_i, i \geq 1\}$ such that $P(N = n) = p(1-p)^{n-1}$, $n = 1, 2, \dots$, $0 < p < 1$. Let $V_N = \max_{1 \leq i \leq N} X_i$. Then $\{V_N\}$ is distributed as MOGuU $(\frac{1}{p}, \lambda, \sigma, \theta)$ if and only if $X_i \sim GuU(\lambda, \sigma, \theta)$.*

Proof. The distribution function of V_N is

$$\begin{aligned}
 L(x) &= P(V_N \leq x) \\
 &= \sum_{n=1}^{\infty} P(V_n \leq x) P(N = n) \\
 &= \sum_{n=1}^{\infty} [G(x)]^n p(1-p)^{n-1} \\
 &= \frac{pG(x)}{1 - (1-p)G(x)} \\
 &= \frac{p\{e^{-\lambda(\frac{\theta}{x}-1)^{1/\sigma}}\}}{1 - \bar{p}\{e^{-\lambda(\frac{\theta}{x}-1)^{1/\sigma}}\}} \tag{7.8}
 \end{aligned}$$

So the survival function is

$$\begin{aligned}
 \bar{L}(x) &= P(V_N > x) \\
 &= 1 - \frac{p\{e^{-\lambda(\frac{\theta}{x}-1)^{1/\sigma}}\}}{1 - \bar{p}\{e^{-\lambda(\frac{\theta}{x}-1)^{1/\sigma}}\}} \\
 &= \frac{\frac{1}{p}\{1 - e^{-\lambda(\frac{\theta}{x}-1)^{1/\sigma}}\}}{1 - \frac{1}{\bar{p}}\{1 - e^{-\lambda(\frac{\theta}{x}-1)^{1/\sigma}}\}} \tag{7.9}
 \end{aligned}$$

which is the survival function of MOGuU $(\frac{1}{p}, \lambda, \sigma, \theta)$. The converse easily follows. $\square$

8 Transmutation Technique in Gumbel - X family of Distributions

An interesting idea of generalizing a distribution, known in literature by transmutation, is derived by using the Quadratic Rank Transmutation Map (QRTM) introduced by Shaw and Buckley (2009). According to them, transmutation maps comprise of the functional composition of the cumulative distribution function of one distribution with

the inverse cumulative distribution (quantile) function of another. Motivated by the need for parametric families of rich and yet tractable distributions in financial mathematics, they used a transmutation map of a non - Gaussian distribution.

According to the Quadratic Rank Transmutation Map (QRTM) approach, a random variable X is said to have transmuted distribution if its cdf is having the general form

$$F(x) = G(x)[(1 + \alpha) - \alpha G(x)] \quad (8.1)$$

where, $G(x)$ is the cdf of the base distribution. Then the pdf is given by

$$f(x) = g(x)[1 + \alpha - 2\alpha G(x)], \quad |\alpha| \leq 1 \quad (8.2)$$

where $g(x)$ is the pdf of the base distribution. If $\alpha = 0$, we have the base distribution.

8.1 The Transmuted Gumbel - Pareto Distribution

Based on the transmuted - G (T - G) family of distributions, we construct a new model called the transmuted Gumbel - Pareto (TrGuP) distribution with cdf

$$F(x) = e^{-\lambda[(\frac{x}{\theta})^k - 1]^{-1/\sigma}} \left[(1 + \alpha) - \alpha e^{-\lambda[(\frac{x}{\theta})^k - 1]^{-1/\sigma}} \right], \quad x \geq \theta \quad (8.3)$$

The pdf corresponding to (8.3) is given by

$$f(x) = \frac{\lambda k}{\sigma \theta} e^{-\lambda[(\frac{x}{\theta})^k - 1]^{-1/\sigma}} [(\frac{x}{\theta})^k - 1]^{-1/\sigma - 1} (\frac{x}{\theta})^{k-1} \left\{ (1 + \alpha) - \alpha e^{-\lambda[(\frac{x}{\theta})^k - 1]^{-1/\sigma}} \right\} \quad (8.4)$$

where θ is the scale parameter and α is the transmuted parameter.

8.2 Gumbel Transmuted X family of Distributions

If we substitute $s(??)$ and $(??)$ in the cdf of Gumbel - X family given in (1.5), we get the cdf of Gumbel transmuted X family and is given by

$$\begin{aligned} G(x) &= \exp \left\{ -\lambda \left[\frac{K(x)(1 + \alpha - \alpha K(x))}{1 - K(x)(1 + \alpha \bar{K}(x))} \right]^{-1/\sigma} \right\} \\ &= \exp \left\{ -\lambda \left[\frac{1 - K(x)(1 + \alpha \bar{K}(x))}{K(x)(1 + \alpha - \alpha K(x))} \right]^{1/\sigma} \right\} \end{aligned} \quad (8.5)$$

The corresponding pdf is given as

$$g(x) = \frac{\lambda}{\sigma} \exp \left\{ -\lambda \left[\frac{1 - K(x)(1 + \alpha \bar{K}(x))}{K(x)(1 + \alpha - \alpha K(x))} \right]^{1/\sigma} \right\} \times \left[\frac{1 - K(x)(1 + \alpha \bar{K}(x))}{K(x)(1 + \alpha - \alpha K(x))} \right]^{1/\sigma - 1} \\ \times \frac{k(x)[1 + \alpha(1 - 2K(x))]}{(K(x)[1 + \alpha - \alpha K(x)])^2} \quad (8.6)$$

We have developed some special cases like Gumbel transmuted Uniform (GTU), Gumbel transmuted Exponential (GTE), Gumbel transmuted Weibull (GTW), Gumbel transmuted Fréchet (GTF), Gumbel transmuted Pareto (GTP) and Gumbel transmuted Lomax (GTL) distributions and studied some of their properties.

9 Data Analysis

The data for breaking stress of carbon fibers of 50 mm length (GPa) given below is obtained from Nicholas and Padgett (2006). This data set was used by Cordeiro and Lemonte (2011) to illustrate the application of the four parameter beta - Birnbaum Saunders distribution when compared to the two parameter Birnbaum Saunders distribution (Birnbaum and Saunders 1969) and Al - Aqtash et al. (2014) to explain the importance of Gumbel - Weibull distribution. Here we compare the newly developed model with Gumbel Uniform distribution introduced by Jose and Joseph (2018). The mles of the model parameters, their standard errors, goodness of fit statistics K - S statistic, AIC, BIC, HQIC, CAIC and p - values are listed in Table 7.

Figure 1 gives the P - P plots of GTU and Gumbel Uniform distributions and estimated cdf is given in Figure 7.14. From these figures and table we can conclude that the newly developed model is a better model as compared to the existing one. The p - value obtained is very high compared to the Gumbel - Uniform distribution in this case.

Table 7: The mles and their standard errors(SE) and the goodness of fit statistics $-\log L$, AIC, CAIC, BIC, HQIC, K-S statistic and p -value for the data set

Model	mles	SE	$-\log L$	AIC	CAIC	BIC	HQIC	K-S	p-value
GTU	$\lambda=1.5966$ $\sigma=0.6705$ $\alpha=0.3386$ $\theta=4.9421$	0.302 0.026 0.077 0.061	111.98	231.96	232.61	231.24	226.04	0.084	0.735
GuU	$\lambda=0.5581$ $\sigma=0.9533$ $\theta=4.9053$	0.151 0.102 0.027	309.48	624.97	625.36	624.43	620.53	0.288	0.0003

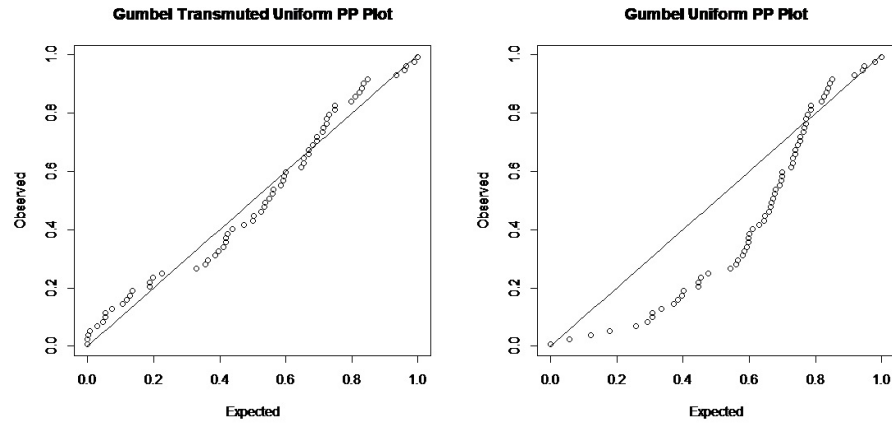


Figure 1: P - P plots for the data set

10 Conclusion

In this paper, we considered Generalized T-X family of distributions and concentrated on the Gumbel - X family of distributions and its special cases. We also explored different properties and applications of these distributions in various areas of statistical theory such as lifetime analysis, industrial reliability test plan, acceptance sampling, time series modeling, stress - strength reliability analysis etc. In addition, we developed transmuted Gumbel - Pareto and Gumbel transmuted X family of distributions. A real data for breaking stress of carbon fibers is used to explain the flexibility of Gumbel Transmuted Uniform model, a member of Gumbel Transmuted X family.

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