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Some Problems Connected with the Distributions of Products and Ratios

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ABSTRACT

In this paper, the distributions of products and ratios involving real scalar positive variables, symmetric products and symmetric ratios of real positive definite matrices and symmetric product and symmetric ratios of Hermitian positive definite matrices in the complex domain are considered. Real scalar variable case is taken as a starting point. Here, it is pointed out how distributions of products and ratios are connected to Bessel integrals, reaction-rate probability integrals in nuclear reaction-rate theory, inverse Gaussian density, Krätzel integral and transform, pathway transform, fractional integrals of the first and second kind etc. Only one illustrative example each is given. Then, symmetric product and symmetric ratios of positive definite matrices in the real and complex domain are considered. Here also one illustrative example each is given. Some current research materials are also mentioned in the introduction.

Key words and Phrases: *Distributions of products and ratios, Bessel integral, reaction-rate probability integral, symmetric product of matrices, symmetric ratio of matrices, fractional integrals of first and second kinds, matrix-variate fractional integrals.*

1 Introduction

Products and ratios of random variables appear in very many places in different fields. One category of problems called scaling models will have the following structure: To start with there is a random variable X, which may be scalar/vector/matrix in the real or complex domain. This is multiplied by a real scalar positive quantity $a > 0$. Then

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$U = aX$ is a scaled X. If the density of X is a function of XX' or $X'X$, where a prime denotes the transpose, then a more appropriate scaling factor is $\sqrt{a}$ so that $U = \sqrt{a}X$. Then, the conditional density of U at given a is available from the density of X by replacing X by $a^{-\frac{1}{2}}U$ with the differential element $\mathrm{d}X$ replaced by $a^{-\frac{pq}{2}}\mathrm{d}U$ if X is a $p \times q$ matrix with pq distinct real scalar variables as elements. If a has its own distribution, then it is a situation of Bayesian analysis with the conditional density given by $f(U|a)$ and a having its own marginal density, say, $g(a)$. Then, the unconditional density is $\int_0^\infty f(U|a)g(a)\mathrm{d}a$. Then, studying the properties of a in the conditional density of a, given U, is the basis in Bayesian analysis. The above is a real scaling model connected to Bayesian analysis. If the scaling factor is a positive definite matrix, then our $U = A^{\frac{1}{2}}X$. Then, in the Bayesian structure, A itself may have a matrix-variate distribution and the conditional density of U, given A, is $f(U|A)$. This situation is very important in current research, especially when X and A are Hermitian positive definite matrices because according the Benavoli et al. (2016), Quantum Physics is nothing but Bayesian analysis of Hermitian positive definite matrices in a Hilbert space. We can treat U as a scaling model by taking A as a constant or having a Bayesian structure with $f(U|A)$ being the conditional density of U, given A, and A having its own distribution. We can treat $U = AX$ as a product of two independently distributed random quantities where each item A and X may be scalar/vector/matrix variable, appropriately defined and having their own distributions. In the same category of problems, we have scalar texture models and matrix texture models in communication and engineering problems. In scalar texture model, A is a real scalar positive quantity and X may be scalar/vector/matrix in the real or complex domain, usually in the complex domain. This is the structure of a cross-section of the mullti-look return signal in radar or sonar. The cross section may have contaminants called speckles and cross section variable itself is called texture. Details of the analysis of multi-look PolSAR (Polarimetric Synthetic Aparture Radar) data is available in Deng (2016). The usual assumption there is that X is Gaussian distributed. But when the surface is not smooth, then non-Gaussian models are preferred, see for example, Frery et al.(1997), Yueh et al. (1989), Bombrun and Bealieu (2008). Recent results are available when X is having a rectangular generalized gamma type density with the exponential trace having an arbitrary power. In this category, there is a model known as Kotz model which has been widely used in Statistics, in texture analysis in communication theory etc. It is shown in Mathai (2023) that the traditionally used normalizing constants were wrong and the correct normalizing constants are given in Mathai (2023). The traditional normalizing constants may be seen from Díaz-Garcia and Gutiérrez Jáimez (2010). In the product model, if A or X has a vector or matrix-variate distribution belonging to logistic type model, then the problem goes into an extended zeta function introduced by Mathai (2023a). Logistic models are preferred compared to standard Gaussian models in industrial applications because the tail probabilities are larger in the logistic models compared to that in the Gaussian model. The above are some of the categories of product models.

This paper is organized as follows: Section 2 starts with the distribution of a product of two real scalar positive random variable, independently distributed. When the densities belong to generalized gamma type, then one has Bessel integrals, reaction-rate probability integrals etc emerging. Section 3 illustrates the connections of densities of products and ratios to fractional integrals and fractional calculus in the real scalar case. Section 4 gives the corresponding connections to matrix-variate fractional calculus in the real and complex domains. Section 5 gives some concluding remarks and points out some open problems.

2 Distribution of a Product

Let us start with two statistically independently distributed real scalar random variables $x_1 > 0, x_2 > 0$. In this paper, real scalar variables whether mathematical variables or random variables, will be denoted by lower-case letters such as x, y. Real vector/matrix variables, mathematical or random, will be denoted by capital letters such as X, Y. Variables in the complex domain will be denoted with a tilde such as $\tilde{x}, \tilde{y}, \tilde{X}, \tilde{Y}$. Scalar constants will be denoted by a, b etc and vector/matrix constants by A, B etc. No tilde will be used on constants. Let the densities associated with x_1 and x_2 be $f_1(x_1)$ and $f_2(x_2)$ respectively. Due to the assumption of independence, the joint density will be $f_1(x_1)f_2(x_2)$. Let us consider the transformation $(x_1, x_2) \to (u = x_1x_2, v = x_2)$. Then, the wedge product of differentials is $\mathrm{d}x_1 \wedge \mathrm{d}x_2 = \frac{1}{v}\mathrm{d}u \wedge \mathrm{d}v$. If the joint density of u and v is denoted by $g(u, v)$, then $g(u, v) = \frac{1}{v}f_1(\frac{u}{v})f_2(v)$ and the marginal density of u, denoted by $g_1(u)$ is given by

$$g_1(u) = \int_v \frac{1}{v} f_1(\frac{u}{v}) f_2(v)\mathrm{d}v = \int_v \frac{1}{v} f_1(v) f_2(\frac{u}{v})\mathrm{d}v. \tag{2.1}$$

The second part of (2.1) is obtained by taking $v = x_1$ instead of taking $v = x_2$. Let us examine (2.1) for some specific densities $f_1(x_1)$ and $f_2(x_2)$. For example, let these be real scalar generalized gamma densities of the form

$$f_j(x_j) = \frac{\delta_j a_j^{\frac{\rho_j}{\delta_j}}}{\Gamma(\frac{\rho_j}{\delta_j})} x_j^{\rho_j - 1} \mathrm{e}^{-a_j x_j^{\delta_j}}, \delta_j > 0, \rho_j > 0, x_j \geq 0, j = 1, 2 \tag{2.2}$$

and zero elsewhere. Let $u = x_1x_2$. Then, from (2.1), the density of u, again denoted by $g_1(u)$, is the following:

$$
\begin{aligned}
g_1(u) &= c_1c_2\int_{v=0}^{\infty}\frac{1}{v}(\frac{u}{v})^{\rho_1-1}\mathrm{e}^{-a_1(\frac{u}{v})^{\delta_1}}v^{\rho_2-1}\mathrm{e}^{-a_2v^{\delta_2}}\mathrm{d}v\\
&= c_1c_2u^{\rho_1-1}\int_{0}^{\infty}v^{\rho_2-\rho_1-1}\mathrm{e}^{-a_2v^{\delta_2}-a_1u^{\delta_1}v^{-\delta_1}}\mathrm{d}v \qquad (2.3)
\end{aligned}
$$

where c_1c_2 is the product of the normalizing constants in (2.2), namely

$$c_1c_2 = \prod_{j=1}^{2}\frac{\delta_j a_j^{\frac{\rho_j}{\delta_j}}}{\Gamma(\frac{\rho_j}{\delta_j})}, \delta_j > 0, \rho_j > 0, a_j > 0, j = 1, 2.$$

The integral in (2.3) is connected to many problems in different areas. For $\delta_2 = 1, \delta_1 = \frac{1}{2}$, the integral in (2.3) is the reaction-rate probability integral in nuclear reaction-rate theory, see Mathai and Haubold (1988). For $\delta_1 = 1, \delta_2 = 1$ the integral in (2.3) is the basic Bessel integral and also for $\delta_j = 1, j = 1, 2$ the integrand in (2.3), normalized, is the inverse Gaussian density in Statistics, for $\delta_2 = 1$ and for a general δ_1 the integral is the Krätzel integral in applied analysis and there is also an integral transform associated with it, known as Krätzel transform. This transform is also extended to cover Mathai's pathway family (see, Mathai, 2005), known as pathway transform. Some authors misinterpreted the integral in (2.3) and called it as generalized gamma, ultra gamma etc. But, it is shown by the author that the integral has no connection to gamma integral and it is connected to Bessel series, Bessel integral etc. Evaluation of the integral in (2.3) is a challenging problem. The author's collaboration with the German group of scientists in the area of Astrophysics started when the German team brought some 12 open problems to the author in 1982. All the problems, when converted from their differential equations format to integral equations, had a basic integral to be solved, which was (2.3) with $\delta_2 = 1$ and $\delta_1 = \frac{1}{2}$. At that time, Mathai could not find any mathematical technique to tackle the integral. Hence, he used statistical techniques to evaluate the general integral in (2.3). Observe that the integrand in (2.3) is a product of positive integrable functions and hence one can create statistical densities out of them. Then, the integral can be identified with the structure in (2.1) and hence the integral could be identified with the density of a product of real scalar positive random variables. Then, the density can be evaluated through some other means, such as through general moments. But the density of a product is unique and hence whatever is obtained as the density of the product is the value of the integral to be evaluated. When $x_1 > 0$ and $x_2 > 0$ are independently distributed, then, denoting the expected value of $(\cdot)$ by $E[(\cdot)]$, we have the following:

$$E[u^{s-1}] = E[x_1^{s-1}]E[x_2^{s-1}], E[x_j^{s-1}] = \int_0^{\infty} x_j^{s-1}f_j(x_j)\mathrm{d}x_j, j = 1, 2 \qquad (2.4)$$

in terms of the $(s-1)$th moments of u, x_1, x_2. Since the variables are positive real scalar variables, one can compare these moments with the Mellin transforms of the functions f_1 and f_2 with Mellin parameter s, denoted by $M_{f_j}(s), j = 1, 2$. Then, from (2.4) we have

$$M_{g_1}(s) = M_{f_1}(s)M_{f_2}(s) \tag{2.5}$$

whenever the Mellin transforms exist or whenever the respective integrals are convergent, where $g_1(u)$ is the density of u. we can also obtain (2.5) by taking the Mellin transform of the left and right sides in (2.1) directly. Once we have the Mellin transform, we can obtain the density g_1 by taking the inverse Melin transform, namely

$$g_1(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M_{f_1}(s)M_{f_2}(s)u^{-s}\mathrm{d}s, i = \sqrt{(-1)} \tag{2.6}$$

which is a contour integral. For applying the Mellin and inverse Mellin transform technique, the functions involved need not be statistical densities. The only requirements are that the Mellin transforms of f_1 and f_2 exist and the product of the Mellin transforms can be inverted. We will evaluate a general integral here so that as particular case, the integral in (2.3) is available. Consider the following integral, denoted by I_w.

$$I_w = \int_0^\infty x^{\gamma-1}\mathrm{e}^{-ax^\delta - b(\frac{w}{x})^\rho}\mathrm{d}x. \tag{2.7}$$

We can identity this integral with the structure

$$I_w = \int_{v=0}^\infty \frac{1}{v}f_1(\frac{w}{v})f_2(v)\mathrm{d}v$$

where $f_1(x) = \mathrm{e}^{-bx^\rho}$ and $f_2(x) = x^\gamma \mathrm{e}^{-ax^\delta}$. Then

$$M_{f_1}(s) = \int_0^\infty x^{s-1}\mathrm{e}^{-bx^\rho}\mathrm{d}x = \frac{1}{\rho}\Gamma(\frac{s}{\rho})b^{-\frac{s}{\rho}}, \Re(s) > 0, b > 0, \rho > 0 \tag{i}$$

where $\Re(\cdot)$ means the real part of $(\cdot)$, and

$$M_{f_2}(s) = \int_0^\infty x^{\gamma+s-1}\mathrm{e}^{-ax^\delta}\mathrm{d}x = \frac{1}{\delta}\Gamma(\frac{\gamma+s}{\delta})a^{-(\frac{\gamma+s}{\delta})}, \Re(s) > -\gamma, \delta > 0. \tag{ii}$$

We have taken all parameters to be real because usually in a statistical problem the parameters are real but the results hold for complex parameter γ also. In this case, the condition is $\Re(\gamma) > 0, \Re(s) > -\Re(\gamma)$. Now, the product of the Mellin transforms from (i) and (ii) is the following:

$$M_{f_1}(s)M_{f_2}(s) = \frac{1}{\delta\rho a^{\frac{\gamma}{\delta}}}\Gamma(\frac{s}{\rho})\Gamma(\frac{\gamma+s}{\delta})(a^{\frac{1}{\delta}}b^{\frac{1}{\rho}})^{-s}. \tag{iii}$$

Hence the corresponding Mellin inverse is our I_w. That is,

$$
\begin{aligned}
I_w &= \frac{1}{\rho\delta a^{\frac{\gamma}{\delta}}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\frac{s}{\rho})\Gamma(\frac{\gamma+s}{\delta})(a^{\frac{1}{\delta}}b^{\frac{1}{\rho}}w)^{-s}\mathrm{d}s \\
&= \frac{1}{\rho\delta a^{\frac{\gamma}{\delta}}}H_{0,2}^{2,0}\left[a^{\frac{1}{\delta}}b^{\frac{1}{\rho}}w\big|_{(0,\frac{1}{\rho}),(\frac{\gamma}{\delta},\frac{1}{\delta})}\right], 0<w<\infty \qquad (2.8)
\end{aligned}
$$

where $H(\cdot)$ is the H-function, where the c in the contour is any positive real number. For the theory and applications of the H-function, see Mathai et al.(2009). The H-function in (2.8) can be put in computable series form. General techniques are available in Mathai (1993). For a special case, this H-function can be written in terms of Bessel hypergeometric series which will be considered here. For $\rho=\delta, \frac{\gamma}{\delta}\neq\pm 0,1,...$ we can write I_w as the following, after replacing $\frac{s}{\delta}$ by s.

$$
I_w = \frac{1}{\delta a^{\frac{\gamma}{\delta}}}\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\Gamma(s)\Gamma(\frac{\gamma}{\delta}+s)(abw^{\delta})^{-s}\mathrm{d}s \qquad (2.9)
$$

where the c_1 in the contour is any positive real number. The value of the contour integral in (2.9) is available as the sum of the residues of the integrand at the poles of $\Gamma(s)$ and at the poles of $\Gamma(\frac{\gamma}{\delta}+s)$. The poles of $\Gamma(s)$ are at the points $s=-n, n=0,1,...$ and the residue at $s=-n$ is given by

$$
\begin{aligned}
\lim_{s\to -n}[(s+n)\Gamma(s)\Gamma(\frac{\gamma}{\delta}+s)(abw^{\delta})^{-s}] &= \lim_{s\to -n}[\frac{(s+n)(s+n-1)...s}{(s+n-1)...s}\Gamma(s)\Gamma(\frac{\gamma}{\delta}+s)(abw^{\delta})^{-s}] \\
&= \lim_{s\to -n}[\frac{\Gamma(s+n+1)}{(s+n-1)...s}\Gamma(\frac{\gamma}{\delta}+s)(abw^{\delta})^{-s}] \\
&= \frac{(-1)^n}{n!}\Gamma(\frac{\gamma}{\delta}-n)(abw^{\delta})^{n}.
\end{aligned}
$$

But, for example,

$$
\Gamma(\alpha-n)=\Gamma(\alpha)\frac{(-1)^n}{(1-\alpha)_n}
$$

where, for example, $(\beta)_n$ is the Pochhammer symbol, $(\beta)_n=\beta(\beta+1)...(\beta+n-1), \beta\neq 0, (\beta)_0=1$. Hence, the sum of the residues of the integrand at the poles of $\Gamma(s)$ is the following:

$$
\Gamma(\frac{\gamma}{\delta})\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{(1-\frac{\gamma}{\delta})_n}(abw^{\delta})^n=\Gamma(\frac{\gamma}{\delta})_0F_1(\ ;1-\frac{\gamma}{\delta};abw^{\delta}).
$$

By proceeding exactly the same way, the sum of the residues of the integrand at the poles of $\Gamma(\frac{\gamma}{\delta}+s)$ is the following:

$$
\Gamma(-\frac{\gamma}{\delta})(abw^{\delta})^{\frac{\gamma}{\delta}}{}_0F_1(\ :1+\frac{\gamma}{\delta};abw^{\delta}).
$$

Hence, from these two series the value of I_w is available. We may state it as a theorem.

Theorem 2.1. *For $a > 0, b > 0, \delta > 0, \rho > 0, \rho = \delta, \frac{\gamma}{\delta} \neq \pm n, n = 0, 1, ...$*

$$
\begin{aligned}
I_w &= \int_0^\infty x^{\gamma-1} \mathrm{e}^{-ax^\delta - b(\frac{w}{x})^\rho} \mathrm{d}x \\
&= \frac{1}{\delta a^{\frac{\gamma}{\delta}}} [\Gamma(\frac{\gamma}{\delta}) {}_0F_1(\ \ ; 1 - \frac{\gamma}{\delta}; abw^\delta) \\
&+ \Gamma(-\frac{\gamma}{\delta})(abw^\delta)^{\frac{\gamma}{\delta}} {}_0F_1(\ \ ; 1 + \frac{\gamma}{\delta}; abw^\delta)], 0 < w < \infty.
\end{aligned}
$$

An evaluation of I_w in terms of Bessel functions of the second kind may be seen from Jeffrey and Zwillinger (2007). Here we have taken f_1 and f_2 as having real scalar variable generalized gamma densities. One can take different types of densities, where the respective Mellin transforms exist, and obtain all types of interesting results.

$$
h(w) = \int_v \frac{1}{v} f_1(\frac{w}{v}) f_2(v) \mathrm{d}v = \int_v \frac{1}{v} f_1(v) f_2(\frac{w}{v}) \mathrm{d}v
$$

with

$$
M_h(s) = M_{f_1}(s) M_{f_2}(s)
$$

is known as the Mellin convolution of a product property. As we have seen that this Mellin convolution of a product property can be directly connected to the density of a product of two real scalar positive random variables. There is also a Mellin convolution of a ratio property. Again, consider two real scalar positive variables $x_1 > 0, x_2 > 0$. Then, we can consider the ratios $\frac{x_1}{x_2}$ or $\frac{x_2}{x_1}$ and for each case when we are transforming to two other variables the second variable v can be taken as x_1 or as x_2. Thus, there are four possibilities when considering a ratio. In the case of a product we had only two possibilities. For example, if $u_2 = \frac{x_2}{x_1}$ and if $g_2(u_2)$ is the function corresponding to u_2, then, we can see that for $v = x_2$, $\mathrm{d}x_1 \wedge \mathrm{d}x_2 = -\frac{v}{u_2^2} \mathrm{d}u_2 \wedge \mathrm{d}v$. Also,

$$
\begin{aligned}
E[u_2^{s-1}] &= E[x_2^{s-1}] E[x_1^{-s+1}] \Rightarrow M_{g_2}(s) = M_{f_2}(s) M_{f_1}(2-s) \\
g_2(u_2) &= \int_v \frac{v}{u_2^2} f_1(\frac{v}{u_2}) f_2(v) \mathrm{d}v. \qquad (2.10)
\end{aligned}
$$

The two results in (2.10) together is called the Mellin convolution of a ratio property and, as seen, it is associated with the density of a ratio when real scalar positive independently distributed random variables are involved. Again, one can obtain a lot of results from this ratio property also by taking different functions f_1 and f_2 or by taking different densities when random variables are involved.

3 Connection of Products and Ratios to Fractional Calculus

In Section 2 we considered a basic problem of deriving the distribution of a product and a ratio of independently distributed real scalar positive variables. One illustrative example given was in terms of both the densities f_1 and f_2 belonging to a generalized gamma family of densities. Now, we consider a slightly different problem. Let $f_1(x_1)$ be a type-1 beta density with the parameters $\gamma+1$ and α or with the density

$$f_1(x_1) = \frac{\Gamma(\gamma+1+\alpha)}{\Gamma(\alpha)\Gamma(\gamma+1)} x_1^{\gamma}(1-x_1)^{\alpha-1}$$

for $0 \le x_1 \le 1, \alpha > 0, \gamma > -1$ and $f_1(x_1) = 0$ elsewhere. When the parameters are complex, the conditions will be $\Re(\alpha) > 0, \Re(\gamma) > -1$. Let $x_2 > 0$ have an arbitrary density $f_2(x_2) = f(x_2)$. Now, let $u = x_1x_2$ the product. Let $g(u)$ be the density of the product u. Then, as shown in Section 2,

$$\begin{aligned} g(u) &= \int_v \frac{1}{v} f_1(\frac{u}{v}) f_2(v)\mathrm{d}v \\ &= \frac{\Gamma(\gamma+1+\alpha)}{\Gamma(\gamma+1)\Gamma(\alpha)} \int_v \frac{1}{v}(\frac{u}{v})^{\gamma}(1-\frac{u}{v})^{\alpha-1} f(v)\mathrm{d}v \\ &= \frac{\Gamma(\gamma+1+\alpha)}{\Gamma(\gamma+1)} K_{2,\gamma}^{-\alpha}(f) \\ K_{2,\gamma}^{-\alpha}(f) &= \frac{u^{\gamma}}{\Gamma(\alpha)} \int_{v\ge u} v^{-\gamma-\alpha}(v-u)^{\alpha-1} f(v)\mathrm{d}v \end{aligned} \tag{3.1}$$

where $K_{2,\gamma}^{-\alpha}(f)$ is Erdélyi-Kober fractional integral of the second kind of order α and parameter γ. Thus, one can interpret a fractional integral of the second kind as a constant multiple of the density of a product of two independently distributed real scalar positive random variables when f_1 and f_2 are densities, with f_1 being a type-1 beta density with the parameters $(\gamma+1, \alpha)$. Note that we have only made a slight change in the densities and then considered the density of a product and we ended up in a fractional integral in fractional calculus. This connection is illustrated in Mathai et al. (2022) and Mathai and Haubold (2017). In a series of papers published in Linear Algebra and Applications, starting from 2013, the author has extended fractional calculus to functions of matrix argument in the real and complex domain, apart from pointing out the connection of fractional integrals to statistical distribution theory. In these papers, a general definition for fractional integrals, encompassing all the various definitions available in the literature, is also given. In the real scalar variable case, the definition will correspond to taking $f_1(x_1) = \frac{1}{\Gamma(\alpha)}\phi(x_1)(1-x_1)^{\alpha-1}$ and $f_2(x_2) = f(x_2)$ an arbitrary function, where f_1 and

f_2 need not be densities, and then taking Mellin convolution of a product. By taking different values of $\phi(x_1)$ one will end up with the various fractional integrals of the second kind available in the literature, such as Riemann-Liouville integral, Weyl integral and so on.

Now, consider the ratio $u_2 = \frac{x_2}{x_1}, x_1 > 0, x_2 > 0$. Let $v = x_2$, then $\mathrm{d}x_1 \wedge \mathrm{d}x_2 = -\frac{v}{u_2^2}\mathrm{d}u_2 \wedge \mathrm{d}v$ and let $g_2(u_2)$ be the function corresponding to u_2. If $x_1 > 0$ and $x_2 > 0$ are independently distributed real scalar random variables, then $g_2(u_2)$ is the density of $u_2 = \frac{x_2}{x_1}$, otherwise we tackle the problem through Mellin convolution of a ratio. Then,

$$g_2(u_2) = \int_v \frac{v}{u_2^2} f_1(\frac{v}{u_2}) f_2(v)\mathrm{d}v. \tag{3.2}$$

Let us take $f_1(x_1)$ to be a type-1 beta density with the parameters $\gamma > 0$ and $\alpha > 0$, that is,

$$f_1(x_1) = \frac{\Gamma(\gamma+\alpha)}{\Gamma(\gamma)\Gamma(\alpha)} x_1^{\gamma-1}(1-x_1)^{\alpha-1}$$

for $0 \le x_1 \le 1, \alpha > 0, \gamma > 0$ and $f_1(x_1) = 0$ elsewhere. Let $f_2(x_2) = f(x_2)$ an arbitrary density. Then,

$$\begin{aligned} g_2(u_2) &= \frac{\Gamma(\gamma+\alpha)}{\Gamma(\gamma)\Gamma(\alpha)} (\frac{v}{u_2})^{\gamma-1}(1-\frac{v}{u_2})^{\alpha-1} f(v)\mathrm{d}v \\ &= \frac{\Gamma(\gamma+\alpha)}{\Gamma(\gamma)} K_{1,\gamma}^{-\alpha}(f) \\ K_{1,\gamma}^{-\alpha}(f) &= \frac{u_2^{-\gamma-\alpha}}{\Gamma(\alpha)} \int_{v \le u_2} v^{\gamma}(u_2 - v)^{\alpha-1} f(v)\mathrm{d}v \end{aligned} \tag{3.3}$$

where $K_{1,\gamma}^{-\alpha}(f)$ is Erdélyi-Kober fractional integral of the first kind of order α and parameter γ. Hence, this fractional integral of the first kind is a constant multiple of a statistical density of a ratio when f_1 and f_2 are statistical densities. From this observation, Mathai introduced a general definition for fractional integral of the first kind by taking $f_1(x_1) = \frac{1}{\Gamma(\alpha)}\psi(x_1)(1-x_1)^{\alpha-1}$ and $f_2(x_2) = f(x_2)$ an arbitrary function and then considering the Mellin convolution of a ratio. It is shown that by specializing $\psi(x_1)$ one can obtain all the different fractional integrals of the first kind available in the literature, where the substitutions are $x = \frac{v}{u_2}$ and $x_2 = v$. Then the idea is extended to symmetric product and symmetric ratio of matrices, which will be discussed in the next section (see, Mathai and Haubold, 2018).

In integer order calculus or the usual calculus we have $D, D^2, \ldots$ where D is the differential operator such as $D = \frac{\mathrm{d}}{\mathrm{d}t}$ if t is the independent variable, whereas in fractional calculus we have the notation for fractional derivative as D^{α} where α can be integer, fractions, negative or positive, or any complex number. Then, anti-integral is taken as fractional derivative. In integer-order calculus, we take the integral as anti-derivative but in fractional calculus we define fractional integral $D^{-\alpha}$ first and then the fractional derivative D^{α} is obtained as an anti-integral. For some integer $n = 1, 2, ...$, let $\Re(n-\alpha) > 0$. Then, consider the fractional integral $D^{-(n-\alpha)}$. Let D^n be the n-th integer order derivative. Then, symbolically we can write $D^{\alpha} = D^n D^{-(n-\alpha)}$ or $D^{\alpha} = D^{-(n-\alpha)} D^n$. The first one is Known as Riemann-Liouville fractional derivative where the arbitrary function f is operated with the fractional integral operator $D^{-(n-\alpha)}$ first, then the n-th derivative is taken. In the second one, known as Caputo derivative, the arbitrary function f is differentiated n times and then a fractional integral $D^{-(n-\alpha)}$ is taken. In both the cases, fractional derivative is a type of integral defined over an interval. In the integer order calculus, a derivative is a local activity, instantaneous rate of change at a given point, whereas in fractional calculus the derivative is defined over an interval which makes it more important in practical applications. In a physical situation, ideal property may be taking place at a given point whereas in a practical situation that property may be taking place either locally at the given point or somewhere in the neighborhood of that point. The ideal point and its neighborhood are covered by a fractional derivative whereas only the ideal point is covered by integer order derivative. This is the physical interpretation given by Mathai as a comparison of fractional versus integer order derivative. From a statistical point of view, one can define and obtain properties of fractional integrals, thereby properties of fractional derivatives, through statistical distribution theory.

4 Symmetric Product and Symmetric Ratio of Matrices

Here we restrict our matrices to be $p \times p$ positive definite when in the real domain and $p \times p$ Hermitian positive definite when in the complex domain. A $p \times p$ matrix $\tilde{X}$ in the complex domain is Hermitian when $\tilde{X} = \tilde{X}^*$, where $\tilde{X}^*$ denotes the conjugate transpose of $\tilde{X}$. Let $X_1 > O$ and $X_2 > O$ be two $p \times p$ real positive definite matrices. Let $f_1(X_1)$ and $f_2(X_2)$ be the functions associated with X_1 and X_2. Let the joint function be $f_1(X_1)f_2(X_2)$ the product. In statistical terms, we are considering two $p \times p$ real matrix-variate random variables with the respective densities $f_1(X_1)$ and $f_2(X_2)$ and we assume that the matrices are statistically independently distributed so that the joint density is the product of the individual densities. Let $U = X_2^{\frac{1}{2}} X_1 X_2^{\frac{1}{2}}$ be the symmetric product of X_1 and X_2 where $X_2^{\frac{1}{2}}$ is the symmetric positive definite square root of the symmetric positive definite matrix $X_2 > O$. We restrict our discussion to positive definite matrices because we will be able to define positive definite square root in a unique way.

If $Y = (y_{ij})$ is a $p \times q$ matrix in the real domain where the elements y_{ij}'s are distinct real scalar variables, then the wedge product of the differentials is denoted by $\mathrm{d}Y = \wedge_{i=1}^{p} \wedge_{j=1}^{q} \mathrm{d}y_{ij}$. If the $p \times p$ matrix $Z = (z_{ij}) = Z'$ (symmetric), where a prime denotes the transpose, then the number of distinct elements possible is only $p(p+1)/2$. Then, $\mathrm{d}Z = \wedge_{i \le j} \mathrm{d}z_{ij} = \wedge_{i \ge j} \mathrm{d}z_{ij}$. If the $p \times q$ matrix $\tilde{Y} = (\tilde{y}_{ij})$ is in the complex domain, then $\tilde{Y}$ can always be written as $\tilde{Y} = Y_1 + iY_2, i = \sqrt{(-1)}, Y_1, Y_2$ are real. Then, the differential element $\mathrm{d}\tilde{Y}$ is defined as $\mathrm{d}\tilde{Y} = \mathrm{d}Y_1 \wedge \mathrm{d}Y_2$. For a $p \times p$ matrix A the determinant will be denoted as $|A|$ or as $\det(A)$. If A is in the complex domain, then $\det(A) = a + ib, i = \sqrt{(-1)}, a, b$ are real scalar quantities. Then, the absolute value of the determinant will be denoted as $|\det(A)| = \sqrt{a^2 + b^2}$. A statistical density $f(X)$ of X is defined as a real-valued scalar function of X such that $f(X) \ge 0$ for all X in the domain of X and $\int_X f(X)\mathrm{d}X = 1$, where X may be scalar or vector or matrix or a collection of matrices, in the real or complex domain.

Let $U = X_2^{\frac{1}{2}} X_1 X_2^{\frac{1}{2}}$ be the symmetric product of $p \times p$ real positive definite matrices as defined above. We can also define a symmetric product as $X_1^{\frac{1}{2}} X_2 X_1^{\frac{1}{2}}$ but the two forms are different. In the real scalar case they are one and the same, but not in the matrix case. Let $V = X_2$. Then, we can show that $\mathrm{d}X_1 \wedge \mathrm{d}X_2 = |V|^{-\frac{p+1}{2}} \mathrm{d}U \wedge \mathrm{d}V$ parallel to that in the scalar case. Note that $V = X_2, U = X_2^{\frac{1}{2}} X_1 X_2^{\frac{1}{2}} \Rightarrow X_2 = V, X_1 = V^{-\frac{1}{2}} U V^{-\frac{1}{2}}$. Then, the marginal density of U, denoted by $g(U)$, is the following:

$$g(U) = \int_V |V|^{-\frac{p+1}{2}} f_1(V^{-\frac{1}{2}} U V^{-\frac{1}{2}}) f_2(V)\mathrm{d}V. \tag{3.4}$$

For matrix transformations and associated Jacobians, see Mathai (1997). If we take f_1 and f_2 as real matrix-variate gamma densities with shape parameters $\alpha_j, j = 1, 2$ and scale parameter matrices as $B_j, j = 1, 2$, then the densities will be of the following forms, where $X_j > O$ is $p \times p$ real positive definite:

$$f_j(X_j) = \frac{|B_j|^{\alpha_j}}{\Gamma_p(\alpha_j)} |X_j|^{\alpha_j - \frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(B_j X_j)} j = 1, 2 \tag{i}$$

where $B_j > O, X_j > O, \Re(\alpha_j) > \frac{p-1}{2}, j = 1, 2$. Note that $B_j > O$ is a constant $p \times p$ real positive definite matix. Here, for example, $\Gamma_p(\alpha)$ is the real matrix-variate gamma, defined as the following:

$$\begin{aligned} \Gamma_p(\alpha) &= \int_{X>O} |X|^{\alpha - \frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X)} \mathrm{d}X, \Re(\alpha) > \frac{p-1}{2} \\ &= \pi^{\frac{p(p-1)}{4}} \Gamma(\alpha)\Gamma(\alpha - \frac{1}{2}) ... \Gamma(\alpha - \frac{p-1}{2}), \Re(\alpha) > \frac{p-1}{2}, \end{aligned} \tag{3.5}$$

where tr$(\cdot)$ means the trace of $(\cdot)$. This $\Gamma_p(\alpha)$ is also known by different names in the literature such as generalized gamma. Mathai called it matrix-variate gamma because it is associated with the matrix-variate gamma integral, given in the first line of (3.5). The corresponding complex matrix-variate gamma function, denoted by $\tilde{\Gamma}_p(\alpha)$ is the following:

$$\begin{aligned}\tilde{\Gamma}_p(\alpha) &= \int_{\tilde{X}>O} |\det(\tilde{X})|^{\alpha-p}\mathrm{e}^{-\mathrm{tr}(\tilde{X})}\mathrm{d}\tilde{X}, \Re(\alpha) > p-1 \\ &= \pi^{\frac{p(p-1)}{2}}\Gamma(\alpha)\Gamma(\alpha-1)...\Gamma(\alpha-p+1), \Re(\alpha) > p-1. \end{aligned} \tag{3.6}$$

If we take f_1 and f_2 as the regular matrix-variate gamma densities from (i) and if we evaluate the density $g(U)$ of U, then we will end up with an integral over V involving a factor of the form $\mathrm{e}^{-\mathrm{tr}(B_2V)-\mathrm{tr}(B_1V^{-\frac{1}{2}}UV^{-\frac{1}{2}})}$ which is a Bessel type or Krätzel type integral in the matrix-variate case, with V and V^{-1} appearing in the exponent. In general, it is an open problem but in some cases it can be evaluated by going through a Fourier transform and then taking the inverse Fourier transform. Let us see whether we can obtain a matrix-variate version of fractional integrals. Let

$$f_1(X_1) = \frac{\Gamma_p(\gamma+\frac{p+1}{2}+\alpha)}{\Gamma_p(\gamma+\frac{p+1}{2})\Gamma_p(\alpha)}|X_1|^{\gamma}|I-X|^{\alpha-\frac{p+1}{2}}$$

and $f_2(X_2) = f(X_2)$ where f is an arbitrary density. Note that

$$\begin{aligned}&|V|^{-\frac{p+1}{2}}|V^{-\frac{1}{2}}UV^{-\frac{1}{2}}|^{\gamma}|I-V^{-\frac{1}{2}}UV^{-\frac{1}{2}}|^{\alpha-\frac{p+1}{2}} \\ &= |U|^{\gamma}|V|^{-\gamma-\alpha}|V-U|^{\alpha-\frac{p+1}{2}}\end{aligned}$$

for $V>U$ in the sense $V-U>O$ (positive definite). Then, the marginal density of U, namely $g(U)$, is the following:

$$\begin{aligned}g(U) &= \frac{\Gamma_p(\gamma+\frac{p+1}{2}+\alpha)}{\Gamma_p(\gamma+\frac{p+1}{2})}K_{2,\gamma}^{-\alpha}(f) \text{ where} \\ K_{2,\gamma}^{-\alpha}(f) &= \frac{|U|^{\gamma}}{\Gamma_p(\alpha)}\int_{V>U}|V|^{-\gamma-\alpha}|V-U|^{\alpha-\frac{p+1}{2}}f(V)\mathrm{d}V\end{aligned}$$

is defined by Mathai as the Erdélyi-Kober fractional integral of the second kind of order α and parameter γ in the real matrix-variate case because for $p=1$ it agrees with such a fractional integral in the real scalar case.

Let us consider one problem in the complex domain also. Let $\tilde{X}$ have a complex matrix-variate type-1 beta density with the parameters $\gamma+p$ and α, namely

$$f_1(\tilde{X}_1) = \frac{\tilde{\Gamma}_p(\gamma+p+\alpha)}{\tilde{\Gamma}_p(\gamma+p)\tilde{\Gamma}_p(\alpha)}|\det(\tilde{X}_1)|^{\gamma}|\det(I-\tilde{X}_1)|^{\alpha-p}$$

for $\Re(\alpha) > p-1, \Re(\gamma) > 0, O < \tilde{X}_1 < I$ in the sense $\tilde{X}_1 > O$ and $I - \tilde{X}_1 > O$ (both Hermitian positive definite) and $f_1(\tilde{X}_1) = 0$ elsewhere. Let $\tilde{X}_2$ have an arbitrary density $f(\tilde{X}_2)$. Let $\tilde{U} = \tilde{X}_2^{\frac{1}{2}} \tilde{X}_1 \tilde{X}_2^{\frac{1}{2}}, V = X_2$. We can show that $\mathrm{d}\tilde{X}_1 \wedge \mathrm{d}\tilde{X}_2 = |\det(V)|^{-p} \mathrm{d}\tilde{U} \wedge \mathrm{d}\tilde{V}$. Then, procedging as in the real case, the marginal density of $\tilde{U}$, denotd by $g(\tilde{U})$, is the following:

$$g(\tilde{U}) = \frac{\tilde{\Gamma}_p(\gamma+p+\alpha)}{\tilde{\Gamma}_p(\gamma+p)} \tilde{K}_{2,\gamma}^{-\alpha}(f)$$

where

$$\tilde{K}_{2,\gamma}^{-\alpha}(f) = \frac{|\det(\tilde{U})|^{\gamma}}{\tilde{\Gamma}_p(\alpha)} \int_{V>U} |\det(V)|^{-\gamma-\alpha} |\det(\tilde{V}-\tilde{U})|^{\alpha-p} f(\tilde{V}) \mathrm{d}\tilde{V}$$

is the fractional integral in the matrix-variate case of the second kind of order α and parameter γ in the complex domain. In a similar manner, we can derive the fractional integral of the first kind of order α and parameter γ in the matrix-variate case in the real and complex domains.

5 Concluding Remarks

Here we have only touched upon the theory of the distributions of products and ratios. We have taken illustrative examples from a real scalar gamma density and type-1 beta density. In the real scalar case, one can connect statistical distribution theory of a product to reaction-rate probability integral, Krätzel integral, inverse Gaussian density, Bessel integral and fractional integrals of the second kind. In the case of a ratio, we have taken only one type of ratio of real scalar positive variables and connected to fractional integral of the first kind. In the ratio case, if we had taken generalized gamma densities then one could have obtained some very interesting integrals, connections to generalized type-2 beta form etc. If we had taken products and ratios of pathway densities, then one could have derived a collection of interesting results. In the matrix-variate case, we have only illustrated connection to fractional integrals by taking a complex matrix-variate type-1 beta density. Here also many other possibilities are there. In the case of matrix-variate gamma densities, we have taken only the standard form where also one could not evaluate the integrals because the integrals go into matrix-variate version of Bessel or Krätzel integral. Matrix-variate gamma densities are also connected to Maxwell-Boltzmann and Raleigh densities in Physics, see Mathai and Princy (2017). There are matrix-variate gamma type densities available with the exponential trace having an arbitrary power or exponent, see Mathai (2023, 2023a). Consideration of the distributions of symmetric products and symmetric ratios of matrices are all open problems when the exponential trace has an arbitrary power.

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