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Cumulative Residual Extropy properties of Ranked Set Sample for Cambanis Type Bivariate Distributions

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ABSTRACT

In this paper, we consider the cumulative residual extropy (CREX) of concomitant of order statistic and its properties, when samples are taken from Cambanis family of distributions. By using the expression for CREX of concomitants of order statistics, the CREX of the RSS observations is obtained, when the study variate Y is difficult to measure but an auxiliary variable X is used to rank the units in each set, under the assumption that (X, Y) follows Cambanis type bivariate distributions.

Key words and Phrases: *Ranked set sampling, Cambanis family of distributions, Concomitants of order statistics, Cumulative residual extropy*

1 Introduction

Let (X, Y) be a bivariate random variable with cumulative distribution function (*cdf*) $F(x, y)$ and joint probability density function (*pdf*) $f(x, y)$. Let $f_X(x)$ and $f_Y(y)$ be the marginal pdfs and $F_X(x)$ and $F_Y(y)$ be the marginal cdfs of X and Y respectively. Let (X_i, Y_i) $i = 1, 2, ..., n$ be a random sample from the bivariate

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population. If we arrange the sample in ascending order of magnitude based on X observations as $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$, then $X_{r:n}$ is the rth order statistic of X_i's. Then the Y variate associated with $X_{r:n}$ is called concomitants of rth order statistic and it is denoted by $Y_{[r:n]}$. The concept of concomitants of order statistics was first introduced by David (1973) . Concomitants of order statistics play an important role in ranked set sampling, double sampling and certain selection procedures. For more details on concomitants of order statistics and its application see, David and Nagaraja (1998).

McIntyre (1952) introduced a new sampling scheme named ranked set sampling (RSS), as an alternative to simple random sampling. Ranked set sampling is applicable whenever ranking of a set of sampling units can be done easily by a judgement method or based on the measurement of an auxiliary variable on the units selected (*see*, *Chen et al.*(2004)). The procedure of RSS is as follows. Choose n^2 units randomly from the population. These units are randomly alloted into n sets, each of size n . Then the units in each set are ranked visually or using some inexpensive methods. From the first set of n units, choose the unit which has the lowest rank for actual measurement. From the second set of n units the unit ranked second lowest is measured. The process is continued until choose the unit which has the highest rank in the nth set. Then make measurement on variable of interest of the selected units, which constitute the ranked set sample(rss).

RSS as described in McIntyre (1952) is applicable whenever sample size is small and ranking of a set of sampling units can be done easily by a judgment method. Suppose the variable of interest, say Y, is expensive to measure and difficult to rank the units. In this case Stoke (1977) modified the method by using an auxiliary variable for ranking the sampling units in each set. Stoke (1977) described the RSS procedure as follows. Choose n^2 units randomly from a bivariate population. Arrange these units into n sets, each of size n and measure the auxiliary variable X. In the first set, that unit for which smallest measurment on the auxiliary variable X is chosen. In the second set, that unit for which second smallest measurment on the auxiliary variable X is chosen. Finally, in the nth set, that unit for which largest measurment on the auxiliary variable X is chosen. Then make measurments of the study variable Y of the selected units and it is denoted by $Y_{[r]}$, r=1,2,...,n. Clearly $Y_{[r]}$ is the concomitant of the rth order statistic of given random sample.

An example for the application of RSS explained by Stokes (1977) is given in Al-Saleh and Al-Ananbeh (2007). Here the study variate Y represents the height of the trees X represents the diameter of trees. Applying the concepts of concomitant of order statistics in ranked set sampling, Chacko and Thomas(2007, 2008, 2009), Chacko (2017) and Tahamsebi and Jafari (2012) estimated the parameters of distribution belonging to Morgenstern family of distributions.

Let X be a random variable with pdf $f_X(x)$ and cdf $F_X(x)$. Then a measure of uncertanity, extropy introduced by Lad et al. (2015) is given by

$$\begin{aligned} J(X) &= \frac{-1}{2}\int_{-\infty}^{\infty}(f_{(X)}(x))^2dx \qquad (1.1)\\ &= \frac{-1}{2}\int_{0}^{1} f_Y(F^{-1}(u))du, \qquad (1.2)\end{aligned}$$

where $F^{-1}(u) = inf\{x; F_X(x) \geq u\}, u \in [0,1]$ is the quantile function of $F_X(x)$. The properties and applications of extropy based on ranked set samples was discussed by Qiu and Eftekharian (2021) .

Jahanshashi et al. (2020) introduced a new measure of information named Cumulative residual extropy (CREX) defined as

$$\begin{aligned} \xi_{\mathcal{J}}(X) &= \frac{-1}{2}\int_{0}^{\infty}(\overline{F}_{(X)}(x))^2dx \qquad (1.3)\\ &= \frac{-1}{2}\int_{0}^{1} \frac{(1-u)^2}{f_X(F^{-1}(u))}du, \qquad (1.4)\end{aligned}$$

where $\overline{F}(x)$ is the survival function of X. Kazemi et al. (2021) discussed the properties of CREX based on minimum rss with unequal samples.

Morgenstern (1956) introduced a family of bivariate distributions which can be constructed with specific marginal distributions. The pdf of (X,Y) which follows a Morgenstern family of distributions (MFD) is given by

$$f(x,y) = f_X(x)f_Y(y)[1+\alpha(2F_X(x)-1)(2F_Y(y)-1)], -1 \leq \alpha \leq 1, \qquad (1.5)$$

where α is the association parameter, $f_X(x)$ and $f_Y(y)$ are the marginal pdfs and $F_X(x)$ and $F_Y(y)$ are the marginal cdfs of X and Y respectively. Cambanis (1977) has modified the Morgenstern family of distributions by introducing new parameters. The cdf of Cambanis type bivariate distributions ($CTBD$), proposed by Cambanis (1977) is given by

$$\begin{aligned} H(x,y) = F_X(x)F_Y(y)[1 + \alpha_1(1 - F_X(x)) + \alpha_2(1 - F_Y(y)) \\ + \alpha_3(1 - F_X(x))(1 - F_Y(y))], \end{aligned} \tag{1.6}$$

where the parameters α_1, α_2 and α_3 are real satisfying the following conditions. $1+\alpha_1+\alpha_2+\alpha_3 > 0$, $1+\alpha_1-\alpha_2-\alpha_3 > 0$, $1-\alpha_1+\alpha_2-\alpha_3 > 0$ and $1-\alpha_1-\alpha_2+\alpha_3 > 0$. The marginal distribution funtions of X and Y are

$$H_X(x) = F_X(x)(1 + \alpha_1(1 - F_X(x)))$$

and

$$H_Y(y) = F_Y(y)(1 + \alpha_2(1 - F_Y(y)))$$

respectively. In this paper, we study the properties of weighted cumulative residual extropy for concomitant of order statistic $Y_{[r:n]}$ and thereby study the properties of extropy of rss when (X,Y) follows Cambanis type bivariate distributions ($CTBD$), proposed by Cambanis (1977) with $\alpha_1 = 0$. The cdf of $CTBD$ with $\alpha_1 = 0$ is given by

$$H(x,y) = F_X(x)F_Y(y)[1 + \alpha_2(1 - F_Y(y)) + \alpha_3(1 - F_X(x))(1 - F_Y(y))], \tag{1.7}$$

where $|\alpha_2 + \alpha_3| < 1$ and $|\alpha_2 - \alpha_3| < 1$ and it is denoted as CTBD(α_2, α_3). Koshti (2021) and Alawady (2021) discussed various properties of Cambanis family of distributions. Chacko and George (2022) discussed the extropy properties of ranked set sample when ranking is not perfect. Chacko and George (2023) explained the extropy properties of ranked set sample for Cambanis family of distributions .

This paper is organized as follows. In section 2, the expresssion for CREX of concomitant of rth order statistic $Y_{[r:n]}$ is obtained. In this section we also obtained upper and lower bounds of CREX of $Y_{[r:n]}$. In section 3, we obtain CREX of ranked set sample arising from $CTBD(\alpha_2, \alpha_3)$ and study its monotonic property . Some bounds for CREX of RSS is given in section 4. In section 5, we give concluding remarks.

2 Cumulative residual extropy of concomitant of rth order statistic

Let $Y_{[r:n]}$ r=1,2,...,n be the concomitant of rth order statistic of a bivariate random sample arising from $CTBD(\alpha_2, \alpha_3)$. Then the pdf of $Y_{[r:n]}$ is given by (see, Thomas and Scaria, 2011)

$$\begin{aligned} f_{Y[r:n]}(y) &= f_Y(y)[1 + (\alpha_2 + \alpha_3 \frac{n - 2r + 1}{n + 1})(1 - 2F_Y(y))] \quad (2.1) \\ &= f_Y(y)[1 + C_{(r,n,\alpha_2,\alpha_3)}(1 - 2F_Y(y))], \quad (2.2) \end{aligned}$$

where

$$C_{(r,n,\alpha_2,\alpha_3)} = \alpha_2 + \alpha_3 \frac{n - 2r + 1}{n + 1}. \quad (2.3)$$

Also the cdf of $Y_{[r:n]}$ is given by

$$F_{Y[r:n]}(y) = F_Y(y)[1 + C_{(r,n,\alpha_2,\alpha_3)}(1 - F_Y(y))]. \quad (2.4)$$

Therefore the survival function of $Y_{[r:n]}$ is given by

$$\overline{F}_{Y_{[r:n]}}(y) = 1 - F_Y(y)[1 + C_{(r,n,\alpha_2,\alpha_3)}(1 - F_Y(y))]. \quad (2.5)$$

Then by using (1.3) the CREX of $Y_{[r:n]}$ is given by

$$\begin{aligned} \xi_{\mathcal{J}}(Y_{[r:n]}) &= \frac{-1}{2} \int_y (\overline{F}_{Y_{[r:n]}}(y))^2 dy \\ &= \frac{-1}{2} \int_y \Big[1 - F_Y(y)\big[1 + C_{(r,n,\alpha_2,\alpha_3)}(1 - F_Y(y))\big]\Big]^2 dy \\ &= \frac{-1}{2} \int_y \Big[\overline{F}_Y(y)[1 - C_{(r,n,\alpha_2,\alpha_3)} F_Y(y)]\Big]^2 dy \\ &= \frac{-1}{2} \int_0^1 \frac{(1 - u)^2 (1 - C_{(r,n,\alpha_2,\alpha_3)} u)^2}{f_Y(F^{-1}(u))} du, \quad (2.6) \\ &= \frac{-1}{2} \int_0^1 \frac{(1 - u)^2 h_{(r,n,\alpha_2,\alpha_3)}(u)^2}{f_Y(F^{-1}(u))} du, \quad (2.7) \end{aligned}$$

where $h_{(r,n,\alpha_2,\alpha_3)}(u) = 1 - C_{(r,n,\alpha_2,\alpha_3)}u$.

Theorem 2.1. *Let $Y_{[r:n]}$ be the concomitant of rth order statistic of a random sample of size n arising from $CTBD(\alpha_2, \alpha_3)$, then the CREX of $Y_{[r:n]}$ can be written as*

$$\xi_{\mathcal{J}}(Y_{[r:n]}) = \xi_{\mathcal{J}}(Y) + \frac{C_{(r,n,\alpha_2,\alpha_3)}}{12} E\left[\frac{1}{f_Y(F^{-1}(U_1))}\right] - \frac{C^2_{(r,n,\alpha_2,\alpha_3)}}{60} E\left[\frac{1}{f_Y(F^{-1}(U_2))}\right], \tag{2.8}$$

where $\xi_{\mathcal{J}}(Y)$ is the CREX of Y and U_i follows $Beta(i+1, 3)$, for $i = 1, 2$ with pdf given by

$$g_i(u) = 2(i+1)(2i+1)u^i(1-u)^2, for\ i = 1, 2. \tag{2.9}$$

Proof. Since $Y_{[r:n]}$ is the concomitant of rth order statistic of a random sample of size n arising from $CTBD(\alpha_2, \alpha_3)$, we have

$$\begin{aligned}(\overline{F}_{Y_{[r:n]}}(y))^2 &= (\overline{F}_Y(y))^2[1 - C_{(r,n,\alpha_2,\alpha_3)}F_Y(y)]^2 \\ &= (\overline{F}_Y(y))^2 - 2C_{(r,n,\alpha_2,\alpha_3)}(\overline{F}_Y(y))^2 F_Y(y) + C^2_{(r,n,\alpha_2,\alpha_3)}(\overline{F}_Y(y))^2(F_Y(y))^2.\end{aligned}$$

Therefore, the CREX of $Y_{[r:n]}$ is given by

$$\begin{aligned}\xi_{\mathcal{J}}(Y_{[r:n]}) &= \frac{-1}{2}\int_y (\overline{F}_{Y_{[r:n]}}(y))^2 dy \\ &= \frac{-1}{2}\Bigg[\int_y (\overline{F}_Y(y))^2 dy - 2C_{(r,n,\alpha_2,\alpha_3)}\int_y (\overline{F}_Y(y))^2 F_Y(y) dy \\ &\quad + C^2_{(r,n,\alpha_2,\alpha_3)}\int_y (\overline{F}_Y(y))^2 (F_Y(y))^2 dy\Bigg] \\ &= I_1 - 2C_{(r,n,\alpha_2,\alpha_3)}I_2 + C^2_{(r,n,\alpha_2,\alpha_3)}I_3,\end{aligned} \tag{2.10}$$

where

$$I_j = \frac{-1}{2}\int_y [\overline{F}_Y(y)]^2[F_Y(y)]^{j-1} dy,\ j = 1, 2, 3.$$

Clearly,

$$\begin{aligned} I_1 &= \frac{-1}{2}\int_y (\overline{F}_Y(y))^2 dy \\ &= \xi_{\mathcal{J}}(Y), \end{aligned}$$

$$\begin{aligned} I_2 &= \frac{-1}{2}\int_y (\overline{F}_Y(y))^2 F_Y(y) dy \\ &= \frac{-1}{2}\int_0^1 \frac{u(1-u)^2}{f_Y(F^{-1}(u))} du \end{aligned}$$

and

$$\begin{aligned} I_3 &= \frac{-1}{2}\int_y (\overline{F}_Y(y))^2 (F_Y(y))^2 dy \\ &= \frac{-1}{2}\int_0^1 \frac{u^2(1-u)^2}{f_Y(F^{-1}(u))} du. \end{aligned}$$

If U_i follows a beta distribution $Beta(i+1,1)$, for $i=1,2$ with pdf given in (2.9) Then we can write

$$I_2 = \frac{-1}{24} E[\frac{1}{f_Y(F^{-1}(U_1))}]$$

and

$$I_3 = \frac{-1}{60} E[\frac{1}{f_Y(F^{-1}(U_2))}].$$

On substituting I_1 ,I_2 and I_3 in (2.10), we get (2.8). Hence the theorem. □

Remark 2.1. *If* $r = 1$ *and* $r = n$ *in* (2.5)*, we get the concomitant of first order statistic and largest order statistic of a random sample of size* n*. Then the CREX of concomitant of first order statistic* $Y_{[1:n]}$ *and concomitant of largest order statistic*

$Y_{[n:n]}$ *are given by*

$$\begin{aligned}\xi_{\mathcal{J}}(Y_{[1:n]}) = \xi_{\mathcal{J}}(Y) &+ \frac{\alpha_2 + \alpha_3 a_n}{12} E\left[\frac{1}{f_Y(F^{-1}(U_1))}\right] \\ &- \frac{(\alpha_2 + \alpha_3 a_n)^2}{60} E\left[\frac{1}{f_Y(F^{-1}(U_2))}\right]\end{aligned}$$

and

$$\begin{aligned}\xi_{\mathcal{J}}(Y_{[n:n]}) = \xi_{\mathcal{J}}(Y) &+ \frac{\alpha_2 - \alpha_3 a_n}{12} E\left[\frac{1}{f_Y(F^{-1}(U_1))}\right] \\ &- \frac{(\alpha_2 - \alpha_3 a_n)^2}{60} E\left[\frac{1}{f_Y(F^{-1}(U_2))}\right],\end{aligned}$$

where $a_n = \frac{n-1}{n+1}$.

Theorem 2.2. *If* $Y_{[r:n]}$ *is the concomitant of* r*th order statistic of a random sample of size* n *arising from* $CTBD(\alpha_2, \alpha_3)$,*then the upper bound of* $\xi_{\mathcal{J}}(Y_{[r:n]})$ *can be written as*

$$\xi_{\mathcal{J}}(Y_{[r:n]}) \leq \frac{C_{(r,n,\alpha_2,\alpha_3)}}{12} E\left[\frac{1}{f_Y(F^{-1}(U_1))}\right], \tag{2.11}$$

where U_1 *follows beta distribution* $Beta(2, 3)$.

Proof. Since $\xi_{\mathcal{J}}(Y) \leq 0$, by using Theorem 2.1 we can obtain the inequality (2.11) directly . Hence the proof. □

Theorem 2.3. *Let* $Y_{[r:n]}$ *be the concomitant of* r*th order statistic of a random sample of size* n *arising from* $CTBD(\alpha_2, \alpha_3)$, *if* $C_{(r,n,\alpha_2,\alpha_3)} \neq 0$, *then the lower bound of* $\xi_{\mathcal{J}}(Y_{[r:n]})$ *is given by*

$$\begin{aligned}\xi_{\mathcal{J}}(Y_{[r:n]}) \geq \frac{-1}{2}&\left(\frac{C^4_{(r,n,\alpha_2,\alpha_3)} - 9C^3_{(r,n,\alpha_2,\alpha_3)} + 36C^2_{(r,n,\alpha_2,\alpha_3)} - 84C_{(r,n,\alpha_2,\alpha_3)} + 126}{630}\right)^{\frac{1}{2}} \\ &\times \left(E\frac{1}{f_Y(y)^2}\right)^{\frac{1}{2}}.\end{aligned} \tag{2.12}$$

Proof. From (2.7), we have

$$\xi_{\mathcal{J}}(Y_{[r:n]}) = \frac{-1}{2}\int_0^1 \frac{(1-u)^2(1 - C_{(r,n,\alpha_2,\alpha_3)}u)^2}{f_Y(F^{-1}(u))} du.$$

By applying Cauchy - Schwarz inequality, we have

$$\xi_{\mathcal{J}}(Y_{[r:n]}) \geq \frac{-1}{2}\left(\int_0^1 (1-u)^4(1-C_{(r,n,\alpha_2,\alpha_3)}u)^4 du\right)^{\frac{1}{2}}\left(\int_0^1 \frac{1}{f_Y(F^{-1}(u))^2}\right)^{\frac{1}{2}}. \quad (2.13)$$

Again

$$\int_0^1 (1-u)^4(1-C_{(r,n,\alpha_2,\alpha_3)}u)^4 du = \frac{C^4_{(r,n,\alpha_2,\alpha_3)} - 9C^3_{(r,n,\alpha_2,\alpha_3)} + 36C^2_{(r,n,\alpha_2,\alpha_3)}}{630} + \frac{-84C_{(r,n,\alpha_2,\alpha_3)} + 126}{630}. \quad (2.14)$$

Also

$$\begin{aligned}\int_0^1 \frac{1}{f_Y(F^{-1}(u))^2} du &= \int \frac{1}{f_Y(y)} dy \\ &= E\frac{1}{f_Y(y)^2}. \end{aligned} \quad (2.15)$$

On substituting (2.14) and (2.15) in (2.13) we get (2.12) . Hence the proof. □

Example 2.1. *If* (X, Y) *follows* $CTBD(\alpha_2, \alpha_3)$ *given in* (1.7) *with marginal pdfs of* X *and* Y *are* $f_X(x) = 1, 0 \leq x \leq 1$ *and* $f_Y(y) = 1, 0 \leq y \leq 1$ *respectively, then*

$$\begin{aligned}\xi_{\mathcal{J}}(Y_{[r:n]}) &= \frac{-1}{2}\int_{u=0}^1 (1-u)^2(1-C_{(r,n,\alpha_2,\alpha_3)}u)^2 du \\ &= \frac{-1}{2}\left[\frac{C^2_{(r,n,\alpha_2,\alpha_3)} - 5C_{(r,n,\alpha_2,\alpha_3)} + 10}{30}\right].\end{aligned}$$

Example 2.2. *If* (X, Y) *follows* $CTBD(\alpha_2, \alpha_3)$ *given in* (1.7) *with marginal pdfs of* X *and* Y *are* $f_X(x) = \theta_1 e^{-\theta_1 x}, x \geq 0$ *and* $f_Y(y) = \theta_2 e^{-\theta_2 y}, y \geq 0$ *respectively , then*

$$\begin{aligned}\xi_{\mathcal{J}}(Y_{[r:n]}) &= \frac{-1}{2}\int_{u=0}^1 \theta_2(1-u)(1-u)^2(1-C_{(r,n,\alpha_2,\alpha_3)}u)^2 du \\ &= \frac{-\theta_2}{2}\left[\frac{C^2_{(r,n,\alpha_2,\alpha_3)} - 6C_{(r,n,\alpha_2,\alpha_3)} + 15}{60}\right].\end{aligned}$$

3 CREX of Ranked Set Sample

Let $Y_{[1]}$,$Y_{[2]}$,...,$Y_{[n]}$ be the ranked set sample of size n arising from $CTBD(\alpha_2, \alpha_3)$ in which X observations are used to rank the units in each set. Clearly $Y_{[r]}$, $r = 1, 2, ..., n$ are the concomitant of order statistic and $Y_{[r]}$, $r = 1, 2, ..., n$ are independent. If $Y_{RSS} = \{Y_{[r]}, r = 1, 2, ..., n\}$,then the CREX of Y_{RSS} can be written as

$$\begin{aligned}
\xi_{\mathcal{J}}(Y_{RSS}) &= \frac{-1}{2}\prod_{r=1}^{n}\int_{y}(\overline{F}_{Y_{[r]}})^2 dy \\
&= \frac{-1}{2}\prod_{r=1}^{n}[-2\xi_{\mathcal{J}}(Y_{[r]})] \\
&= (-1)^{n-1}2^{n-1}\prod_{r=1}^{n}\Bigg[\xi_J(Y) + \frac{C_{(r,n,\alpha_2,\alpha_3)}}{12}E\Bigg[\frac{1}{f_Y(F^{-1}(U_1))}\Bigg] \\
&\quad - \frac{C^2_{(r,n,\alpha_2,\alpha_3)}}{60}E\Bigg[\frac{1}{f_Y(F^{-1}(U_2))}\Bigg]\Bigg].
\end{aligned}$$

Example 3.1. *If* (X, Y) *follows* $CTBD(\alpha_2, \alpha_3)$ *given in* (1.7) *with marginal pdfs of* X *and* Y *are* $f_X(x) = 1, 0 \le x \le 1$ *and* $f_Y(y) = 1, 0 \le y \le 1$ *respectively , then*

$$\xi_{\mathcal{J}}(Y_{RSS}) = \frac{-1}{2}\prod_{r=1}^{n}\left[\frac{C^2_{(r,n,\alpha_2,\alpha_3)} - 5C_{(r,n,\alpha_2,\alpha_3)} + 10}{30}\right].$$

Example 3.2. *If* (X, Y) *follows* $CTBD(\alpha_2, \alpha_3)$ *given in* (1.7) *with marginal pdfs of* X *and* Y *are* $f_X(x) = \theta_1 e^{-\theta_1 x}, x \ge 0$ *and* $f_Y(y) = \theta_2 e^{-\theta_2 y}, y \ge 0$ *respectively , then*

$$\xi_{\mathcal{J}}(Y_{RSS}) = \tfrac{-\theta_2^n}{2}\textstyle\prod_{r=1}^{n}\left[\frac{C^2_{(r,n,\alpha_2,\alpha_3)} - 6C_{(r,n,\alpha_2,\alpha_3)} + 15}{60}\right].$$

Definition 3.1. *(Shaked and Shanthikumar (2007)) Let* X_1 *and* X_2 *be two random variables with cdfs* F_1 *and* F_2 *and pdfs* f_1 *and* f_2 *respectively. The left continuous inverses of* F_1 *and* F_2 *are given by* $F_1^{-1}(u) = inf\{t : F_1(t) \ge u\}$ *and* $F_2^{-1}(u) = inf\{t : F_2(t) \ge u\}$, $0 \le u \le 1$. *Then* X_1 *is said to be smaller than* X_2 *in dispersive order denoted by* $X_1 \le_{disp} X_2$ *if* $F_2^{-1}(F_1(x)) - x$ *is increasing in* $x \ge 0$. *Clearly if* $X_1 \le_{disp} X_2$ *, then* $f_1(F_1^{-1}(u)) \le f_2(F_2^{-1}(u))$, *for* $0 \le u \le 1$.

Theorem 3.1. *Let (X, Y) follows $CTBD(\alpha_2, \alpha_3)$ given in (1.5) with marginal cdfs $F_X(x)$ and $F_Y(y)$ and pdfs $f_X(x)$ and $f_Y(y)$ respectively. Let $Y_{RSS} = \{Y_{[r]}, r = 1, 2, ..., n\}$ be the ranked set sample of size n arising from $CTBD(\alpha_2, \alpha_3)$ in which X observations are used to rank the units. Let (V, W) be another bivariate random variable follows $CTBD(\alpha_2, \alpha_3)$ given in (1.3) with marginal cdfs $G_V(v)$and $G_W(w)$ and pdfs $g_V(v)$ and $g_W(w)$ respectively. Let $W_{RSS} = \{W_{[r]}, r = 1, 2, ..., n\}$ be ranked set sample of size n arising from (V, W) in which observations on $V$are used to rank the units. If $Y \leq_{disp} W$, then $\xi_{\mathcal{J}}(Y_{RSS}) \geq \xi_{\mathcal{J}}(W_{RSS})$.*

Proof. We have

$$\xi_{\mathcal{J}}(Y_{RSS}) \quad = \quad \frac{-1}{2}\prod_{r=1}^{n}\int_0^1 \frac{(1-u)^2(1-C_{(r,n,\alpha_2,\alpha_3)}u)^2}{f_Y(F^{-1}(u))}du.$$

If $Y \leq_{disp} W$, we have $f_Y(F^{-1}(u)) \geq g_W(G^{-1}(u))$ for all u in $(0, 1)$.
Therefore

$$\begin{aligned}\xi_{\mathcal{J}}(Y_{RSS}) \quad &\geq \quad \frac{-1}{2}\prod_{r=1}^{n}\int_0^1 \frac{(1-u)^2(1-C_{(r,n,\alpha_2,\alpha_3)}u)^2}{g_W(G^{-1}(u))}du \\ &= \quad \xi_{\mathcal{J}}(W_{RSS}).\end{aligned}$$

Hence the proof. □

4 Bounds of $\xi_{\mathcal{J}}(Y_{RSS})$

In this section, we obtain some lower bounds and upper bounds for $\xi_{\mathcal{J}}(Y_{RSS})$. Before that we give some properties of $h_{(r,n,\alpha_2,\alpha_3)}(u)$ given in (2.6). We have tabulated the value of $h_{(r,n,\alpha_2,\alpha_3)}(u)$ for $n = 10$, $r = 1, 2, ..., 10$ and for the different combinations of (α_2, α_3) as $(-0.2, -0.7), (-0.2, 0.7), (0.2, -0.7)$ and $(0.2, 0.7)$ and are given in Table 1 and Table 2. We also drawn the graphs of $h_{(r,n,\alpha_2,\alpha_3)}(u)$ against u for different combinations of (α_2, α_3) given in Figure 1 to Figure 4 and the graphs of $h_{(r,n,\alpha_2,\alpha_3)}(u)$ against r for different combinations of (α_2, α_3) given in Figure 5 to Figure 8. Clearly the value of $h_{(r,n,\alpha_2,\alpha_3)}(u)$ lies between 0 and 2.

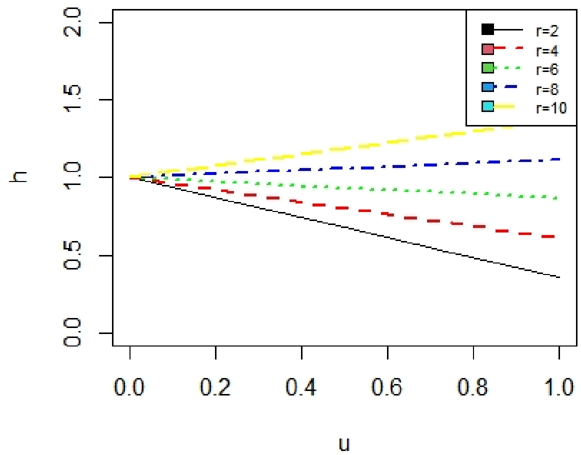

Figure 1: Graph of $h_{(r,n,\alpha_2,\alpha_3)}(u)$ against u when $\alpha_2 = 0.2$ and $\alpha_3 = 0.7$ for $n = 10$

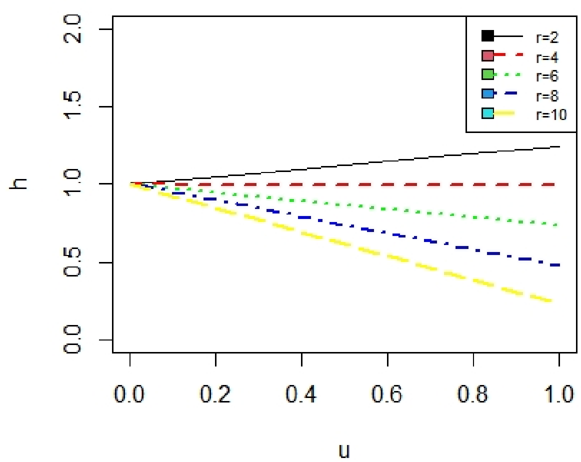

Figure 2: Graph of $h_{(r,n,\alpha_2,\alpha_3)}(u)$ against u when $\alpha_2 = 0.2$ and $\alpha_3 = -0.7$ for $n = 10$

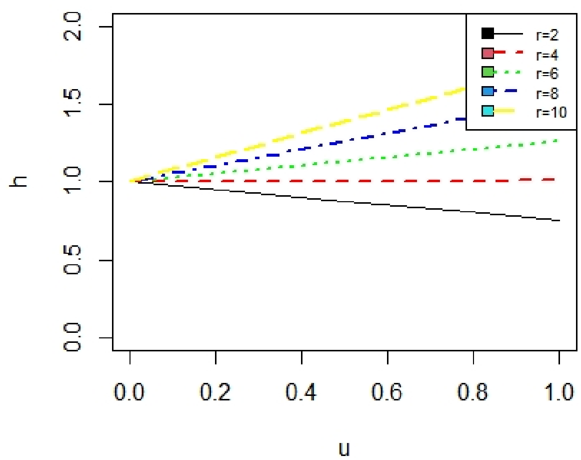

Figure 3: Graph of $h_{(r,n,\alpha_2,\alpha_3)}(u)$ against u when $\alpha_2 = -0.2$ and $\alpha_3 = 0.7$ for $n = 10$

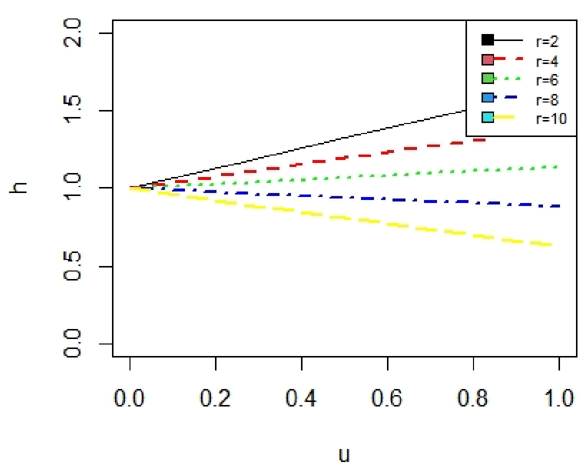

Figure 4: Graph of $h_{(r,n,\alpha_2,\alpha_3)}(u)$ against u when $\alpha_2 = -0.2$ and $\alpha_3 = -0.7$ for $n = 10$

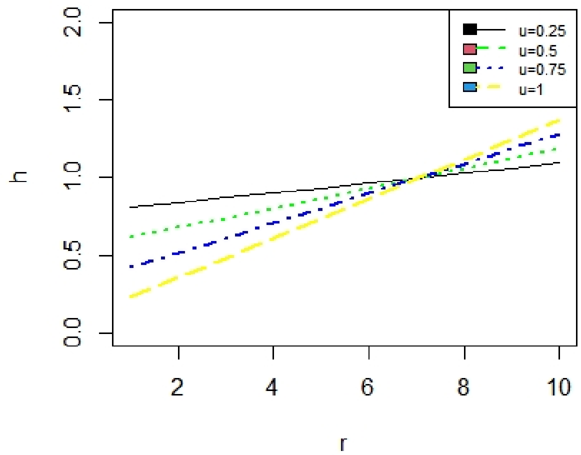

Figure 5: Graph of $h_{(r,n,\alpha_2,\alpha_3)}(u)$ against r when $\alpha_2 = 0.2$ and $\alpha_3 = 0.7$ for $n = 10$

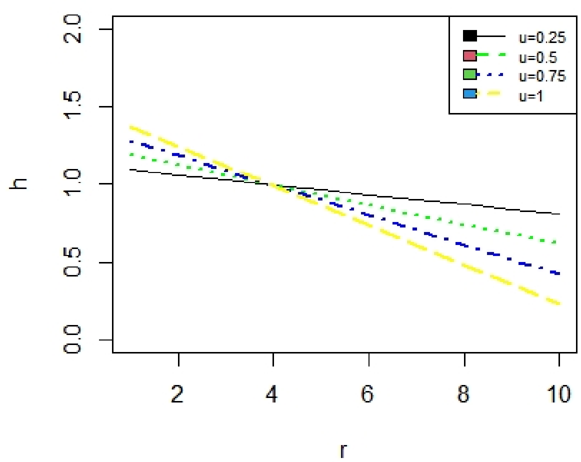

Figure 6: Graph of $h_{(r,n,\alpha_2,\alpha_3)}(u)$ against r when $\alpha_2 = 0.2$ and $\alpha_3 = -0.7$ for $n = 10$

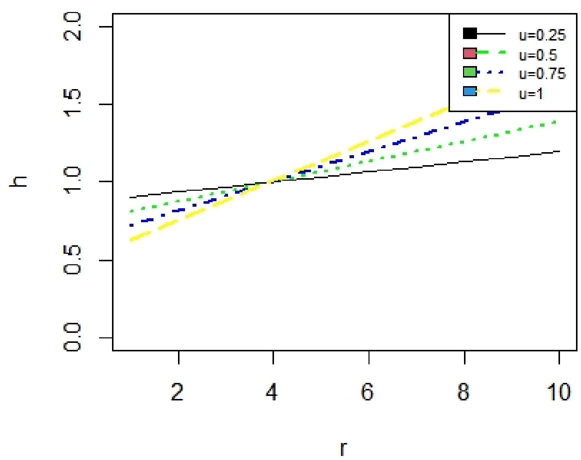

Figure 7: Graph of $h_{(r,n,\alpha_2,\alpha_3)}(u)$ against r when $\alpha_2 = -0.2$ and $\alpha_3 = 0.7$ for $n = 10$

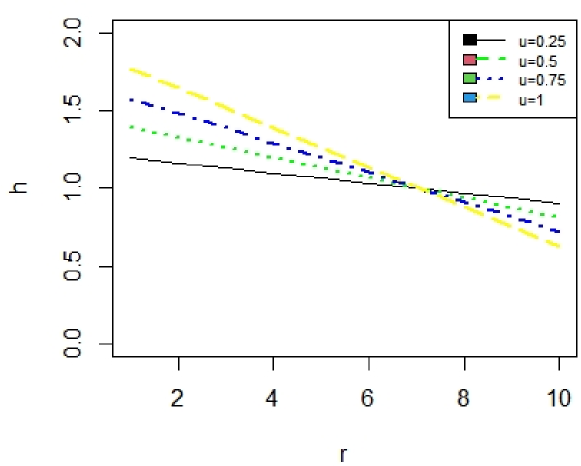

Figure 8: Graph of $h_{(r,n,\alpha_2,\alpha_3)}(u)$ against r when $\alpha_2 = -0.2$ and $\alpha_3 = -0.7$ for $n = 10$

Table 1: $h_{(r,n,\alpha_2,\alpha_3)}(u)$ when α_3 is positive for $n = 10$.

	α_2=0.2 and α_3=0.7				α_2=-0.2 and α_3=0.7			
r	u=0.25	u=0.5	u=0.75	u=1	u=0.25	u=0.5	u=0.75	u=1
1	0.8068	0.6136	0.4205	0.2273	0.9068	0.8136	0.7205	0.6273
2	0.8386	0.6773	0.5159	0.3545	0.9386	0.8773	0.8159	0.7545
3	0.8705	0.7409	0.6114	0.4818	0.9705	0.9409	0.9114	0.8818
4	0.9023	0.8045	0.7068	0.6091	1.0023	1.0045	1.0068	1.0091
5	0.9341	0.8682	0.8023	0.7364	1.0341	1.0682	1.1023	1.1364
6	0.9659	0.9318	0.8977	0.8636	1.0659	1.1318	1.1977	1.2636
7	0.9977	0.9955	0.9932	0.9909	1.0977	1.1955	1.2932	1.3909
8	1.0295	1.00591	1.0886	1.1182	1.1295	1.2591	1.3886	1.5182
9	1.0614	1.1227	1.1841	1.2455	1.1614	1.3227	1.4841	1.6455
10	1.0932	1.1864	1.2795	1.3727	1.1932	1.3864	1.5795	1.7727

Table 2: $h_{(r,n,\alpha_2,\alpha_3)}(u)$ when α_3 is negative for $n = 10$.

	α_2=0.2 and α_3=-0.7				α_2=-0.2 and α_3=-0.7			
r	u=0.25	u=0.5	u=0.75	u=1	u=0.25	u=0.5	u=0.75	u=1
1	1.0932	1.1864	1.2795	1.3727	1.1932	1.3864	1.5795	1.7727
2	1.0614	1.1227	1.1841	1.2455	1.1614	1.3227	1.4841	1.6455
3	1.0295	1.0591	1.0886	1.1182	1.1295	1.2591	1.3886	1.5182
4	0.9977	0.9955	0.9932	0.9909	1.0977	1.1955	1.2932	1.3909
5	0.9659	0.9318	0.8977	0.8636	1.0659	1.1318	1.1977	1.2636
6	0.9341	0.8682	0.8023	0.7364	1.0341	1.0682	1.1023	1.1364
7	0.9023	0.8045	0.7068	0.6091	1.0023	1.0045	1.0068	1.0091
8	0.8705	0.7409	0.6114	0.4818	0.9705	0.9409	0.9114	0.8818
9	0.8386	0.6773	0.5159	0.3545	0.9386	0.8773	0.8159	0.7545
10	0.8068	0.6136	0.4205	0.2273	0.9068	0.8136	0.7205	0.6273

Theorem 4.1. *Let $Y_1, Y_2, .., Y_n$ be a simple random sample from a distribution with cdf $F_Y(y)$ and pdf $f_X(x)$.Let $\{Y_{[r]}, r = 1, 2, ..., n\}$ be the ranked set sample of size n arising from $CTBD(\alpha_2, \alpha_3)$ in which X observations are used to rank the units in each . If $Y_{SRS} = \{Y_1, Y_2, .., Y_n\}$ and $Y_{RSS} = \{Y_{[r]}, r = 1, 2, ..., n\}$, then for $n \geq 1$,*

$$\frac{\xi_{\mathcal{J}}(Y_{RSS})}{\xi_{\mathcal{J}}(Y_{SRS})} \leq \prod_{r=1}^{n}(h_{(r,n,\alpha_2,\alpha_3)}(u_0))^2, \tag{4.1}$$

where u_0 is the value of u which maximise $h_{(r,n,\alpha_2,\alpha_3)}(u) = 1 - C_{(r,n,\alpha_2,\alpha_3)}u$.

Proof. We have

$$\begin{aligned}\xi_{\mathcal{J}}(Y_{SRS}) &= \frac{-1}{2}\prod_{r=1}^{n}\int_y(\overline{F}_Y(y))^2 dy \\ &= \frac{-1}{2}\prod_{r=1}^{n}\int_0^1 \frac{(1-u)^2}{f_Y(F^{-1}(u))}du.\end{aligned}$$

Then,

$$\xi_{\mathcal{J}}(Y_{RSS}) = \frac{-1}{2}\prod_{r=1}^{n}\int_0^1 \frac{(1-u)^2}{f_Y(F^{-1}(u))}h^2_{(r,n,\alpha_2,\alpha_3)}(u)du.$$

Let u_0 be the value of u which maximise $h_{(r,n,\alpha_2,\alpha_3)}(u)$. Then,

$$\begin{aligned}\xi_{\mathcal{J}}(Y_{RSS}) &\geq \frac{-1}{2}\prod_{r=1}^{n}\int_0^1\left(\frac{(1-u)^2}{f_Y(F^{-1}(u))}(h_{(r,n,\alpha_2,\alpha_3)}(u_0))^2\right)du \\ &= \frac{-1}{2}\prod_{r=1}^{n}\left(\int_0^1 \frac{(1-u)^2}{f_Y(F^{-1}(u))}du\right)\prod_{r=1}^{n}(h_{(r,n,\alpha_2,\alpha_3)}(u_0))^2 \\ &= \xi_{\mathcal{J}}(Y_{SRS})\prod_{r=1}^{n}(h_{(r,n,\alpha_2,\alpha_3)}(u_0))^2.\end{aligned}$$

Since $\xi_{\mathcal{J}}(Y_{SRS}) < 0$

$$\frac{\xi_{\mathcal{J}}(Y_{RSS})}{\xi_{\mathcal{J}}(Y_{SRS})} \leq \prod_{r=1}^{n}(h_{(r,n,\alpha_2,\alpha_3)}(u_0))^2.$$

Hence the proof. $\square$

Theorem 4.2. *Let $Y_{RSS} = \{Y_{[r]}, r = 1, 2, ..., n\}$ be the ranked set sample of size n arising from $CTBD(\alpha_2, \alpha_3)$ in which X observations are used to rank the units in each set. Then for $n \geq 1$, the lower bound of CREX of Y_{RSS} is given by*

$$\xi_{\mathcal{J}}(Y_{RSS}) \geq \frac{-1}{2}\Big(E\frac{1}{(f_Y(y))^2}\Big)^{\frac{n}{2}} \times \prod_{r=1}^{n}\left(\frac{C^4_{(r,n,\alpha_2,\alpha_3)} - 9C^3_{(r,n,\alpha_2,\alpha_3)} + 36C^2_{(r,n,\alpha_2,\alpha_3)} - 84C_{(r,n,\alpha_2,\alpha_3)} + 126}{630}\right)^{\frac{1}{2}}.$$

Proof. We have

$$\xi_{\mathcal{J}}(Y_{RSS}) = \frac{-1}{2}\prod_{r=1}^{n}\int_0^1 \frac{(1-u)^2(1-C_{(r,n,\alpha_2,\alpha_3)}u)^2}{f_Y(F^{-1}(u))}du.$$

Using Cauchy-Schwarz inequality

$$\begin{aligned}\xi_{\mathcal{J}}(Y_{RSS}) &\geq \frac{-1}{2}\prod_{r=1}^{n}\Big(\int_0^1 \frac{1}{\big(f_Y(F^{-1}(u))\big)^2}du\Big)^{\frac{1}{2}}\Big(\int_0^1 (1-u)^2(1-C_{(r,n,\alpha_2,\alpha_3)}u)^2du\Big)^{\frac{1}{2}}\\ &= \frac{-1}{2}\Big(E\frac{1}{(f_Y(y))^2}\Big)^{\frac{n}{2}}\\ &\quad\times \prod_{r=1}^{n}\left(\frac{C^4_{(r,n,\alpha_2,\alpha_3)} - 9C^3_{(r,n,\alpha_2,\alpha_3)} + 36C^2_{(r,n,\alpha_2,\alpha_3)} - 84C_{(r,n,\alpha_2,\alpha_3)} + 126}{630}\right)^{\frac{1}{2}}.\end{aligned}$$

Hence the proof. □

5 Conclusion

In this work, we considered the information content of ranked set sample for Cambanis type bivariate distributions using CREX measure when ranking is subject to error. If we consider a ranked set sampling in which an auxiliary variable is used to rank the units in each set, then the observation of RSS are independently distributed concomitants of order statistics. In this work, we derived the CREX of concomitants of order statistics and then derived the CREX of RSS in which the ranking of the units are based on measurments of easily and exactly measurable auxiliary variable X which is correlated with the study variable Y , under the assumption that (X, Y) follows $CTBD(\alpha_2, \alpha_3)$.

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