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Asymmetric Double Lomax Distribution: Theory and Applications

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ABSTRACT

The double Lomax distribution is the ratio of two independent and identically distributed classical Laplace distributions and it can be used to analyse data sets with heavy tails and peakedness. In the present paper, we introduce asymmetric generalization of double Lomax distribution. We derived the probability density function of asymmetric double Lomax distribution and its various properties were studied. The maximum likelihood estimation procedure is employed to estimate the parameters of the proposed distribution and an algorithm in R package is developed to carry out the estimation. To validate the algorithm, simulation studies were conducted with various parameter values. Finally, we fitted the asymmetric double Lomax, asymmetric Laplace, double Lomax and Gaussian distributions to microarray gene expression dataset and financial datasets and compared them.

Key words and Phrases: *Asymmetric double Lomax distribution , Asymmetric Laplace distribution , Double Lomax distribution , Laplace distribution , Microarray gene expression.*

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1 Introduction

There has recently been a surge of interest in constructing more flexible distributions that can account for symmetry, skewness, kurtosis, and heavier tails in data. In the fields of biology, economics, finance, psychology, and sociology, the response variables and hence the errors in regression models do not have to follow Gaussian law. In fact, in many cases, the response variable distributions have heavy tails. As a result, more flexible parametric distributions are required to handle error structures in these circumstances. Skew-symmetric distributions have gained a lot of attention in the statistical literature during the last two decades. One reason for this is that many of the existing symmetric families of distributions are unable to describe skewed data sets.

Laplace distributions and their asymmetric generalisations have recently found use in a variety of fields, including life testing, microarray modelling, and telecommunications. Asymmetric Laplace (see, Kotz et al., 2001) is an asymmetric version of the Laplace distribution that can be used to model asymmetric and heavy-tailed data sets. Many generalizations of Laplace distribution were studied so far. Esscher transformed Laplace (ETL) distribution (see, Sebastian and Dais, 2012) is another asymmetric generalizations of Laplace distribution and the first Order Moving Average Model with Esscher Transformed Laplace innovations was introduced in Bindu and Dais (2017). Recently Punathumparambath (2020) introduced Transmuted Esscher transformed Laplace ($TETL$) distribution. Arezoomand et al. (2018) introduced new asymmetric generalization of Laplace distribution called asymmetric Uniform-Laplace (AUL) distribution. Kumar and Jose (2019) proposed an alternative version to the Laplace distribution called alternative Laplace distribution (ALD). Punathumparambath and Kulathinal (2015) introduced double Lomax distribution (DLD) which is the ratio of two independent and identically distributed Laplace distributions. In this article we present an asymmetric generalization of double Lomax distribution (see, Punathumparambath and Kulathinal, 2015) called the asymmetric double Lomax distribution.

Fernandez and Steel (1998) proposed a method to convert symmetric distribution into an asymmetric distribution by introducing an inverse scale factors in the positive and negative orthants of the symmetric distribution. Symmetric double Lomax distribution is introduced by Punathumparambath and Kulathinal (2015). We extended this distribution by applying the method introduced by Fernandez and Steel (1998) and propose a family of asymmetric double Lomax (ADL) distributions. The double Lomax (DL) distribution is a natural symmetric extension of Lomax distribution (see, Lomax 1958) to the real line and is the ratio of two independent and identically distributed classical Laplace distributions (see, Kotz et al. 2001) and is defined as follows.

Definition 1.1. *Let X_1 and X_2 be two independent and identically distributed (i.i.d.) standard classical Laplace random variables. Then the corresponding probability distribution of $X = X_1/X_2$ is given by*

$$f(x) = \frac{1}{2\,(1+|x|)^2}, \quad -\infty < x < \infty. \tag{1.1}$$

The Laplace distribution can also be expressed as the difference of two i.i.d exponentials and hence, $X_i \stackrel{d}{=} I_i E_i$, for $i = 1, 2$ where, $E_i \sim Exp(1)$, for $i = 1,\ 2$ and $I_i's$ are independent of E_i and takes values ± 1 with equal probabilities.
Then the cdf can be given by

$$F(x) = \begin{cases} \frac{1}{2(1-x)}, & \text{for } x \leq 0, \\ \\ \left(1 - \frac{1}{2(1+x)}\right), & \text{for } x > 0. \end{cases} \tag{1.2}$$

Rest of this article is organized as follows. In Section 2, we define the ADL probability density function (pdf), cumulative distribution function (cdf) and reliability measures. In Section 3, important mathematical properties of the ADL are derived such as, quantile function, moments, an expression for the Rényi entropy and order statistics. Maximum likelihood estimation of parameters using the BFGS algorithm of `optim` function (see, Nash, 1990 and R Core Team, 2020) are described in Section 4. Simulation studies were carried out to illustrate the performance of the algorithm and is presented in Section 5. Applications of the proposed distributions to microarray gene expression dataset and three financial datasets were illustrated in Section 6. Finally some concluding remarks were given in Section 7.

2 Asymmetric Double Lomax Distribution

Using the Fernandez and Steel (see, Fernandez and Steel, 1998) approach, we introduce asymmetry into the symmetric double Lomax distribution. To convert a symmetric distribution into an asymmetric distribution, inverse scale factors are introduced in the positive and negative orthants. As a result, a symmetric density f produces the skewed distributions indexed by $\kappa > 0$.

If $g(\cdot)$ is symmetric on $\Re$, then for any $\kappa > 0$, a skewed density can be obtained as

$$f(x) = \frac{2\kappa}{1+\kappa^2} \begin{cases} g(x\kappa), & \text{for } x > 0, \\ g(\frac{x}{\kappa}), & \text{for } x \leq 0. \end{cases} \tag{2.1}$$

In Eq. (2.1), when g is the symmetric double Lomax distribution with density (1.1), we get a skewed distribution with the density function defined as follows.

Definition 2.1. *A random variable X follows an Asymmetric Double Lomax (ADL) distribution with parameters (μ, σ, k), denoted by $X \sim ADL(\mu, \sigma, \kappa)$ if its probability density function is given by*

$$f(x) = \frac{\kappa}{\sigma(1+\kappa^2)} \begin{cases} (1 - \frac{1}{\sigma\ \kappa}(x-\mu))^{-2}, & \textit{for } x \leq \mu, \\ (1 + \frac{\kappa}{\sigma}(x-\mu))^{-2}, & \textit{for } x > \mu, \end{cases} \tag{2.2}$$

and $\mu \in \mathbb{R}$, $\sigma, \kappa > 0$.

The parameters (μ, σ, κ) are the location, scale, and skewness parameters, respectively.

The cumulative distribution function (cdf) of the ADL distribution is given by

$$F(x) = \begin{cases} \frac{\kappa^2}{1+\kappa^2}[1 - \frac{1}{\sigma\ \kappa}(x-\mu)]^{-1}, & \text{for } x \leq \mu, \\ \\ 1 - \frac{1}{1+\kappa^2}[1 + \frac{\kappa}{\sigma}(x-\mu)]^{-1}, & \text{for } x > \mu. \end{cases} \tag{2.3}$$

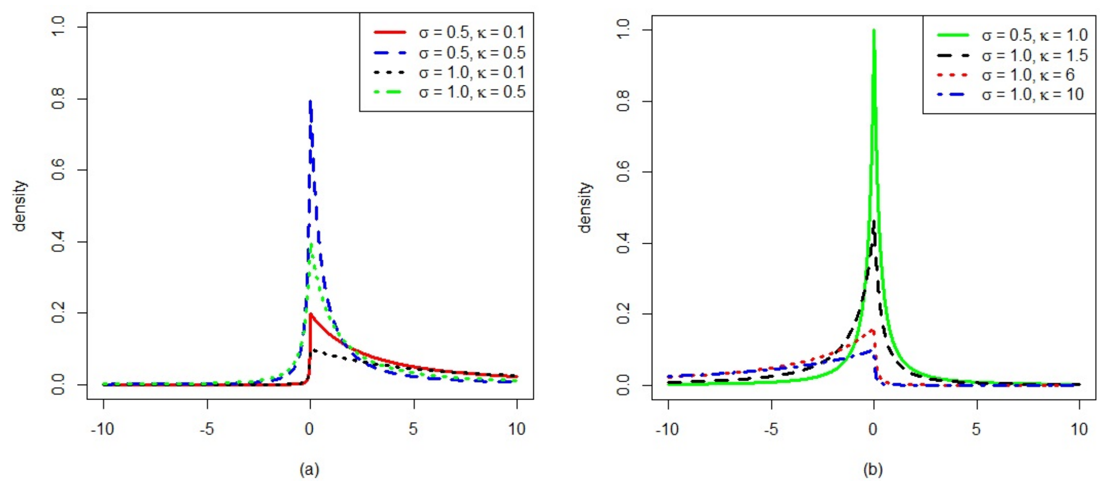


Figure 1: Plots of density of ADL for $\mu = 0$ and for various values of σ and κ, (a) $0 < \kappa < 1$ (b) $\kappa \geq 1$.

When $\kappa = 1$ we get the symmetric double Lomax distribution.

Plots of the pdf of ADL given in Eq. (2.2) for $\mu = 0$ and for various values of σ and κ are given below.

Figure 1 presents the plots of the pdf of $ADL(\mu, \sigma, \kappa)$ for $0 \leq \kappa < 1$ and $\kappa \geq 1$. From Figure 1 we can see that ADL is asymmetric and unimodal.

Now we derive expressions for the reliability measures such as survival function, hazard rate function, reverse hazard rate function and odds function for the ADL so that they can be useful for studying reliability of a system involving one unit.

The survival function (*sf*) of the ADL distribution is given by

$$S(x) = \begin{cases} 1 - \frac{\kappa^2}{1+\kappa^2}[1 - \frac{1}{\kappa\sigma}(x-\mu)]^{-1}, & \text{for } x \leq \mu, \\ \\ \frac{1}{1+\kappa^2}[1 + \frac{\kappa}{\sigma}(x-\mu)]^{-1}, & \text{for } x > \mu. \end{cases} \tag{2.4}$$

Graph of *cdf* given in Eq. (2.3) and *sf* presented in Eq. (2.4) of ADL for various values of σ and κ are given in Figure 2.

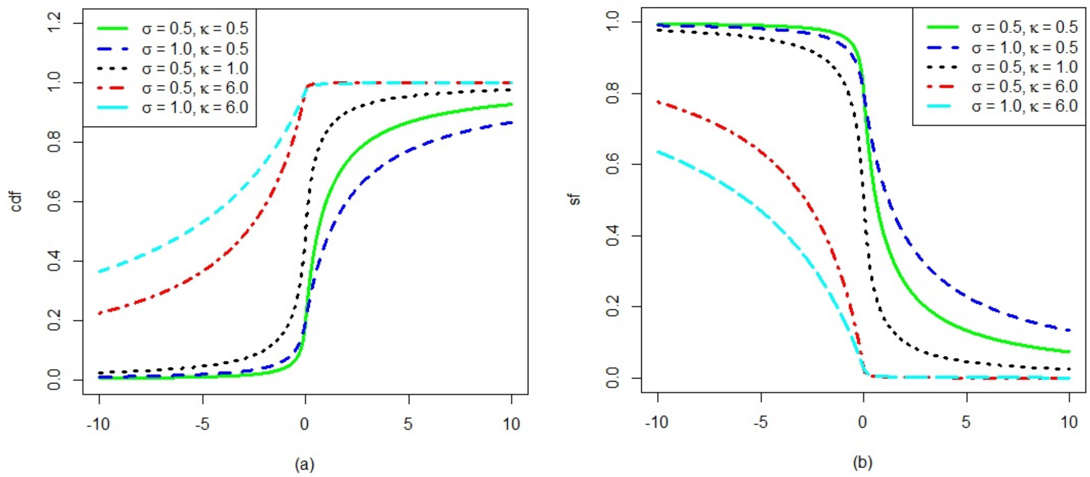


Figure 2: (a) Plots of cumulative distribution function (cdf) and (b) plots of survival function (sf) of ADL for $\mu = 0$ and various values of parameters σ and κ.

The hazard function (*hf*) of the ADL distribution is given by

$$h(x) = \begin{cases} \frac{\kappa}{\sigma(1+\kappa^2)} \frac{[1-\frac{1}{\sigma\ \kappa}(x-\mu)]^{-2}}{1-\frac{\kappa^2}{1+\kappa^2}\left(1-\frac{1}{\sigma\ \kappa}(x-\mu)\right)^{-1}}, & \text{for } x \le \mu, \\ \\ \frac{\kappa}{\sigma}[1 + \frac{\kappa}{\sigma}(x-\mu)]^{-1}, & \text{for } x > \mu. \end{cases} \tag{2.5}$$

Graph of hazard function (*hf*) given in Eq. (2.5) of ADL for various values of σ and κ are given in Figure 3. From the Figure 3 we can observe that failure rate of ADL exhibits both increasing and decreasing behaviour over the real line. The reversed hazard function (*rhf*) of the ADL distribution, $r(x) = \frac{f(x)}{F(x)}$ is given by

$$r(x) = \begin{cases} \frac{1}{\sigma\kappa}[1 - \frac{1}{\sigma\ \kappa}(x-\mu)]^{-1}, & \text{for } x \le \mu, \\ \\ \frac{\kappa}{\sigma(1+\kappa^2)} \frac{[1+\frac{\kappa}{\sigma}(x-\mu)]^{-2}}{1-\frac{1}{1+\kappa^2}[1-\frac{1}{\sigma\ \kappa}(x-\mu)]^{-1}}, & \text{for } x > \mu. \end{cases} \tag{2.6}$$

Graph of reversed hazard function (*rhf*) of ADL for various values of σ and κ are given in Figure 4. From the Figure 4 we can observe that reverse failure rate of ADL exhibits both increasing and decreasing behaviour over the real line.

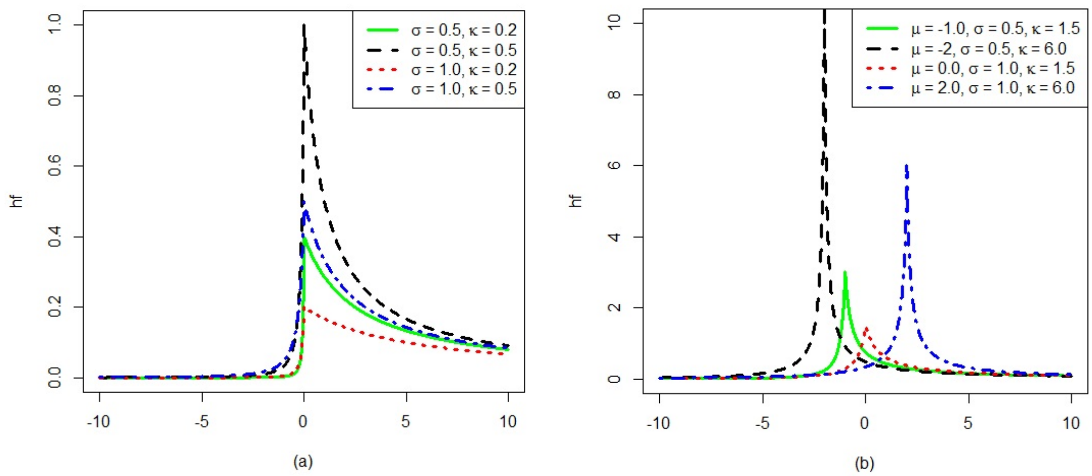


Figure 3: Plots of hazard function (hf) of ADL for various values of parameters μ, σ and κ and (a) $\mu = 0$ and $0 < \kappa < 1$ (b) $\kappa \geq 1$.

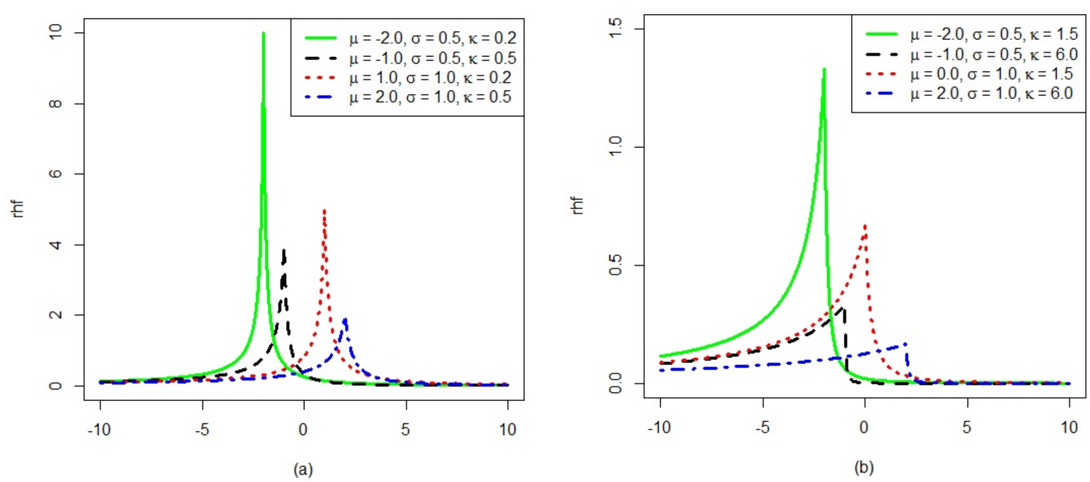


Figure 4: Plots of reversed hazard function (rhf) of ADL for various values of parameters μ, σ and κ, (a) $0 < \kappa < 1$ (b) $\kappa \geq 1$.

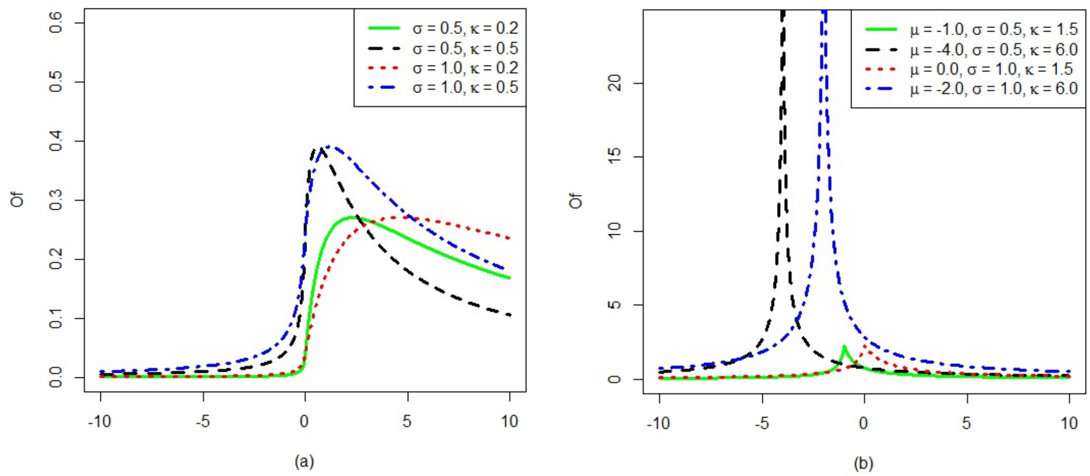


Figure 5: Plots of Odds function (Of) of ADL for various values of parameter μ, σ and κ and (a) $\mu = 0$ and $0 < \kappa < 1$ (b) $\kappa \geq 1$.

The odds function (of) of the ADL distribution $O(x) = \frac{F(x)}{S(x)}$, is given by

$$O(x) = \begin{cases} \dfrac{\frac{\kappa^2}{1+\kappa^2}[1-\frac{1}{\sigma\ \kappa}(x-\mu)]^{-1}}{1-\frac{\kappa^2}{1+\kappa^2}[1-\frac{1}{\kappa\sigma}(x-\mu)]^{-1}}, & x < \mu, \\ \dfrac{1-\frac{\kappa^2}{1+\kappa^2}[1+\frac{\kappa}{\sigma}(x-\mu)]^{-1}}{\frac{\kappa^2}{1+\kappa^2}[1+\frac{\kappa}{\sigma}(x-\mu)]^{-1}}, & x \geq \mu. \end{cases} \tag{2.7}$$

Graph of odds function (of) of ADL for various values of σ and κ are given in Figure 5.

3 Mathematical Properties

In this section, we provide some main mathematical properties of ADL distribution. Figure 1 shows density plots of ADL distributions for various values of κ. For ADL distribution both tails are power tails and the rate of convergence depends on the values of κ. When $\kappa < 1$ the curve moves to the right of the symmetric curve giving heavier right tail, and vice versa when $\kappa > 1$. Right tail becomes heavier as κ approaching to zero and left tail becomes heavier as the value of κ increasing from one. For $\kappa = 1$, the distribution is symmetric.

Below we list a few important properties of ADL distributions. We refer Kotz et al. (2001) for properties that are similar to asymmetric Laplace (AL).

(i) Double Lomax (DL) is the ratio of two independent and identically distributed classical Laplace distributions. Hence we can represent asymmetric double Lomax random variable, $X \sim ADL(\mu, \sigma, \kappa)$, as follows:

$X \stackrel{d}{=} \mu + \frac{I_1 E_1}{I_2 E_2} = \mu + I \frac{E_1}{E_2}$ where, $E_i's$, $\; i = 1,\ 2$, are exponentially distributed with means σ/κ and $\sigma\kappa$, respectively. I take values ± 1 with equal probabilities and is independent of $E_i's$.

(ii) If $X \sim ADL(\mu, \sigma, \kappa)$, then

$$P(X \leq \mu) = \frac{\kappa^2}{1+\kappa^2} = q_\kappa$$

$$P(X > \mu) = \frac{1}{1+\kappa^2} = p_\kappa.$$

The parameter κ controls the probability assigned to each side of μ and for $\kappa = 1$, the two probabilities are equal and the distribution is symmetric about μ.

(iii) The $ADL(\mu, \sigma, \kappa)$ density can be written as a mixture of two Lomax densites with parameters s_1 and s_2 respectively and is as follows:

$$f(x) = p_\kappa \frac{1}{s_1}\left(1 + \frac{(x-\mu)}{s_1}\right)^{-2} I_{(\mu,\ \infty)}(x) + q_\kappa \frac{1}{|s_2|}\left(1 + \frac{(x-\mu)}{s_2}\right)^{-2} I_{(-\infty,\ \mu)}(x),$$

were $s_1 = \sigma/\kappa$, $s_2 = -\sigma\kappa$, $I_A(x)$ is the indicator function, which is equal to 1 if x belong to the set A and zero otherwise. p_κ and q_κ are defined in property (ii).

(iv) If $X \sim ADL(\mu, \sigma, \kappa)$, then $Y = aX + b \sim ADL(b + a\mu, |a|\sigma, \kappa_a)$, where $a \in \mathbb{R}$, $a \neq 0$, $b \in \mathbb{R}$ and $\kappa_a = \kappa^{sign(a)}$, $sign(a) = 1$, if $a > 0$, $sign(a) = -1$, if $a < 0$ and $sign(a) = 0$, if $a = 0$. Hence, the distribution of a linear combination of a random variable with $ADL(\mu, \sigma, \kappa)$ distribution is also ADL. If $X \sim ADL(\mu, \sigma, \kappa)$, then $Y = \frac{(X-\mu)}{\sigma} \sim ADL(0, 1, \kappa)$, which can be called as the standard ADL distribution.

(v) The mode of the distribution is μ and the value of the density function at μ is $(\kappa/(1+\kappa^2))/\sigma$.

(vi) The value of the distribution function at μ is $\kappa^2/(1+\kappa^2)$ and hence, μ is also the $\kappa^2/(1+\kappa^2)$-quantile of the distribution.

(vii) ADL distributions have heavier tails than AL distributions. Note that the tail probability of the symmetric double Lomax density is $\bar{F} \sim cx^{-1}, \;\; as \;\; x \to \pm\infty$, where $\bar{F}$ is the survival function. The heavy tail characteristic makes this densities appropriate for modeling network delays, signals and noise, financial risk or microarray gene expression datasets.

3.1 Quantile Function

The quantile function of X can be obtained by inverting cdf (2.3) and is given in Eq. (3.1). From cdf (2.3), the q^{th} quantile function (qf) of ADL distribution is,

$$\xi_q = \begin{cases} \mu + \kappa\sigma \left[1 - \frac{1}{\left[q\frac{1+\kappa^2}{\kappa^2}\right]}\right], & \text{for } \; q \in \left(0, \frac{\kappa^2}{1+\kappa^2}\right], \\ \\ \mu + \frac{\sigma}{\kappa}\left[\frac{1}{((1-q)(1+\kappa^2))} - 1\right], & \text{for } \; q \in \left(\frac{\kappa^2}{1+\kappa^2}, 1\right). \end{cases} \tag{3.1}$$

The median of ADL is obtained by putting $q = 1/2$ in (3.1). For $q = \kappa^2/(1+\kappa^2)$, the q^{th} quantile is given by $\xi_q = \mu$. Hence, for given κ the location parameter is given by $\hat{\mu} = \xi_{[\kappa^2/(1+\kappa^2)]}$.

The *cdf* and qf can be useful for goodness-of-fit, simulation purposes and computing measures of skewness and kurtosis. The skewness and kurtosis can be defined based on the quantile function. The Galtons skewness (Galton (Galton (1883))) and the Moors kurtosis (Moors (Moors (1988))) coefficients are, respectively

$$S = \frac{\xi_{[6/8]} - 2\xi_{[4/8]} + \xi_{[2/8]}}{\xi_{[6/8]} - \xi_{[2/8]}},$$

$$K = \frac{\xi_{[7/8]} - \xi_{[5/8]} + \xi_{[3/8]} - \xi_{[1/8]}}{\xi_{[6/8]} - \xi_{[2/8]}}.$$

The distribution is symmetric, right (or left) skewed for $S = 0$, $S > 0$ (*or* $S < 0$), respectively. As the value of kurtosis increases, the tail heaviness of the distribution increases.

3.2 Moments

In this subsection we derive the r^{th} moment of ADL distribution. Let $X \sim ADL(\mu, \sigma, \kappa)$ then the r^{th} moment, $\mu_r = E(X-\mu)^r = E(Y^r)$, $0 < r < 1$ is given by

$$\begin{aligned} \mu_r &= \frac{\kappa}{\sigma(1+\kappa^2)} \left(\int_{-\infty}^{0} y^r \left(1 - \frac{y}{\sigma\kappa}\right)^{-2} \mathrm{dx} + \int_0^{\infty} \mathrm{y}^{\mathrm{r}} \left(1 + \frac{\kappa \mathrm{y}}{\sigma}\right)^{-2} \mathrm{dx} \right), \\ &= \frac{\sigma^r}{\kappa^r(1+\kappa^2)} \left(1 + (-1)^r \kappa^{2(r+1)}\right) B(r+1, 1-r), 0 < r < 1. \end{aligned} \tag{3.2}$$

Where $B(a, b)$ is a beta function. It is clear that the moments of order 1 or greater do not exist. Now we derive the r^{th} absolute moment of ADL.
For $0 < r < 1$, the r^{th} absolute moment of ADL is given by

$$\begin{aligned} \delta_r &= \frac{\kappa}{\sigma(1+\kappa^2)} \left(\int_{-\infty}^{0} (-y)^r \left(1 - \frac{y}{\sigma\kappa}\right)^{-2} \mathrm{dx} + \int_0^{\infty} \mathrm{y}^{\mathrm{r}} \left(1 + \frac{\kappa \mathrm{y}}{\sigma}\right)^{-2} \mathrm{dx} \right), \\ &= \frac{\sigma^r}{\kappa^r(1+\kappa^2)} \left(1 + \kappa^{2(r+1)}\right) B(r+1, 1-r), 0 < r < 1. \end{aligned} \tag{3.3}$$

In the next section we derive expressions for the entropy measure of the ADL.

3.3 Entropy

In this subsection we discuss the Rényi entropy of ADL distribution. Entropies quantify the diversity, uncertainty, or randomness of a system and they can be useful for studying index diversity in Ecology. Also it is used as a measure of entanglement in quantum information.

An entropy of a random variable X is a measure of variation of the uncertainty. A popular entropy measure is Rényi entropy (Rényi (1961)). Rényi entropy is defined as follows

$$H_\alpha = \frac{1}{1-\alpha} \log\left(\int f^\alpha(x)\, \mathrm{dx} \right), \ \alpha > 0, \alpha \neq 1.$$

Now we derive Rényi entropy measure of ADL.

$$H_\alpha = \frac{1}{1-\alpha} \log\left(\frac{\kappa^\alpha}{\sigma^\alpha(1+\kappa^2)^\alpha} \left(\int_{-\infty}^{\mu} [1 - \frac{1}{\sigma\ \kappa}(x-\mu)]^{-2\alpha}\, \mathrm{dx} + \int_{\mu}^{\infty} [1 + \frac{\kappa}{\sigma}(\mathrm{x}-\mu)]^{-2\alpha}\, \mathrm{dx} \right) \right).$$

Consider

$$\begin{aligned}\int f^{\alpha}(x)\ \mathrm{dx} &= \frac{\kappa^{\alpha}}{\sigma^{\alpha}(1+\kappa^2)^{\alpha}}\left(\frac{\sigma\kappa}{(2\alpha-1)}+\frac{\sigma}{\kappa(2\alpha-1)}\right),\\ &= \frac{1}{(2\alpha-1)}\left(\frac{\kappa}{\sigma(1+\kappa^2)}\right)^{\alpha-1}.\end{aligned}$$

Then

$$H_{\alpha} = \log\ \sigma+\log\ (1+\kappa^2)-\log\ \kappa-\frac{1}{1-\alpha}\log\ (2\alpha-1). \tag{3.4}$$

3.4 Order Statistics

For asymmetric distributions the measures of interest are median, quartiles, etc. Order statistics are used to estimate these quantities. Here we give the distribution of order statistics arising from ADL random sample. Let $X_1,\cdots,X_n$ is a random sample from ADL distribution and $X_{1:n},\cdots,X_{n:n}$, denote the corresponding order statistics. Then the probability density function of the r^{th} order statistics is given by

$$f_{r:n}(x)=\frac{f(x,\mu,\sigma,\kappa)}{B(r,n-r+1)}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}(F(x))^{r+i-1}, \tag{3.5}$$

Inserting pdf (2.2) and cdf (2.3) in (3.5), we get

$$\begin{aligned}f_{r:n}(x) &= \frac{n!}{(r-1)!(n-r)!}\frac{1}{\sigma\kappa}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}\left(\frac{\kappa^2}{1+\kappa^2}\right)^{r+i}\left(1-\frac{x-\mu}{\sigma\kappa}\right)^{-(r+i+1)},x<\mu,\\ f_{r:n}(x) &= \frac{n!}{(r-1)!(n-r)!}\frac{\kappa}{\sigma}\sum_{i=0}^{n-r}\sum_{j=0}^{r+i-1}(-1)^{i+j}\binom{n-r}{i}\binom{r+i-1}{j}\frac{1}{(1+\kappa^2)^{r+i}}\\ &\quad\times\left(1+\frac{\kappa(x-\mu)}{\sigma}\right)^{-(r+i+1)},x\geq\mu.\end{aligned}$$

The c.d.f of the r^{th} order statistics is given by

$$\begin{aligned}F_{r:n}(x) &= \sum_{i=r}^{n}\binom{n}{i}(F(x))^i(1-F(x))^{n-i},\\ &= \sum_{i=r}^{n}\sum_{j=0}^{n-i}(-1)^j\binom{n}{i}\binom{n-i}{j}(F(x))^{i+j}, \end{aligned} \tag{3.6}$$

Inserting cdf (2.2) in (3.7), we get

$$F_{r:n}(x)=\begin{cases}\displaystyle\sum_{i=r}^{n}\sum_{j=0}^{n-i}(-1)^j\binom{n}{i}\binom{n-i}{j}\left(\frac{\kappa^2}{1+\kappa^2}\right)^{i+j}\left(1-\frac{x-\mu}{\sigma\kappa}\right)^{-(i+j)}, & x<\mu,\\ \displaystyle\sum_{i=r}^{n}\sum_{j=0}^{n-i}(-1)^j\binom{n}{i}\binom{n-i}{j}\left(1-\frac{1}{1+\kappa^2}\left(1+\frac{\kappa}{\sigma}(x-\mu)\right)^{-1}\right)^{i+j}, & x\geq\mu.\end{cases} \tag{3.7}$$

In particular, the pdf of the smallest order statistics $X_{1:n}$ is obtained from (3.6), by substituting $r=1$, as follows

$$f_{1:n}(x)=\frac{nk}{\sigma(1+\kappa^2)}\begin{cases}\left(1-\frac{\kappa^2}{1+\kappa^2}\left(1-\frac{x-\mu}{\sigma\kappa}\right)^{-1}\right)^{n-1}\left(1-\frac{x-\mu}{\sigma\kappa}\right)^{-2}, & x<\mu,\\ \frac{1}{(1+\kappa^2)^{n-1}}\left(1+\frac{\kappa}{\sigma}(x-\mu)\right)^{-(n+1)}, & x\geq\mu.\end{cases} \tag{3.8}$$

Also, the pdf of the largest order statistics $X_{n:n}$ is obtained from (3.6), by substituting $r=n$, as follows

$$f_{n:n}(x)=\frac{nk}{\sigma(1+\kappa^2)}\begin{cases}\left(\frac{\kappa^2}{1+\kappa^2}\right)^{n-1}\left(1-\frac{x-\mu}{\sigma\kappa}\right)^{-(n+1)}, & x<\mu,\\ \left(1-\frac{1}{1+\kappa^2}[1+\frac{\kappa}{\sigma}(x-\mu)]^{-1}\right)^{n-1}[1+\frac{\kappa}{\sigma}(x-\mu)]^{-2}, & x\geq\mu.\end{cases} \tag{3.9}$$

Also the cdf of the smallest order statistics $X_{1:n}$ is obtained from (3.8), by substituting $r=1$, as follows

$$F_{1:n}(x)=\begin{cases}1-\left(1-\frac{\kappa^2}{1+\kappa^2}[1-\frac{x-\mu}{\sigma\kappa}]^{-1}\right)^{n}, & x<\mu,\\ 1-\frac{1}{(1+\kappa^2)^n}[1+\frac{\kappa}{\sigma}(x-\mu)]^{-n}, & x\geq\mu.\end{cases} \tag{3.10}$$

Also the cdf of the largest order statistics $X_{n:n}$ is obtained from (3.8), by substituting $r=n$, as follows,

$$F_{n:n}(x)=\begin{cases}\left(\frac{\kappa^2}{1+\kappa^2}\right)^{n}[1-\frac{x-\mu}{\sigma\kappa}]^{-n}, & x<\mu,\\ \left(1-\frac{1}{1+\kappa^2}[1+\frac{\kappa(x-\mu)}{\sigma}]^{-1}\right)^{n}, & x\geq\mu.\end{cases} \tag{3.11}$$

3.4.1 Moments of Order Statistic

In this section we derive the k^{th} moment of r^{th} order statistic $X_{r:n}$ of ADL. q^{th} moment of r^{th} order statistic $Y_{r:n}=X_{r:n}-\mu$ is defined as

$$E(Y_{r:n}^q)=\int_{-\infty}^{\infty}y^q f_{r:n}(y)\ \mathrm{dy},$$

$$\begin{aligned}
E(Y_{r:n}^q) &= \frac{n!}{\sigma\kappa(r-1)!(n-r)!}\int_{-\infty}^{0} y^q \sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}\left(\frac{\kappa^2}{1+\kappa^2}\right)^{r+i}\left(1-\frac{y}{\sigma\kappa}\right)^{-(r+i+1)}\mathrm{dx} \\
&\quad +\frac{n!}{(r-1)!(n-r)!}\frac{\kappa}{\sigma}\int_{-\infty}^{0} x^q \sum_{i=0}^{n-r}\sum_{j=0}^{r+i-1}(-1)^{i+j}\binom{n-r}{i}\binom{r+i-1}{j}\frac{1}{(1+\kappa^2)^{r+i}} \\
&\quad \times\left(1+\frac{\kappa y}{\sigma}\right)^{-(r+i+1)}\mathrm{dx}, \\
&= \frac{n!}{(r-1)!(n-r)!}(\sigma\kappa)^q\sum_{i=0}^{n-r}(-1)^{i+q}\left(\frac{\kappa^2}{1+\kappa^2}\right)^{r+i} B(q+1,r+i-q) \\
&\quad +\frac{n!}{(r-1)!(n-r)!}\left(\frac{\sigma}{\kappa}\right)^q\sum_{j=0}^{n-r}\sum_{n=0}^{r+i-1}(-1)^{i+j}\binom{n-r}{i}\binom{r+i-1}{j} \\
&\quad \times\frac{1}{(1+\kappa^2)^{r+i}}B(q+1,r+i-q), r+i-q>0.
\end{aligned}$$

4 Estimation

In this section we study the problem of estimating three unknown parameters, $\Theta = (\mu, \sigma, \kappa)$, of the *ADL* distribution. To estimate the parameter μ we use the quantile estimation. Let $X = (X_1, \cdots, X_n)$ be independent and identically distributed samples from an *ADL* distribution with parameters Θ. The quantile estimate of μ is given by $\hat{\mu} = \xi_{[\kappa^2/(1+\kappa^2)]}$. Given κ, the quantile estimator of μ is the sample quantile of order $\kappa^2/(1+\kappa^2)$, which is (for large n) the $([[n\kappa^2/(1+\kappa^2)]]+1)^{th}$ ordered observation, ($[[c]]$ denoted the integral part of c). When the data are approximately symmetric the estimate of μ will be close to the median. The method of maximum likelihood estimation can be employed to estimate Θ as described below. To estimate Θ using maximum likelihood estimation where the likelihood function is maximised to estimate the unknown parameters. We describe this method briefly as follows.

The log-likelihood function of the data X takes the form

$$logL(\Theta; X) = n\log\kappa - n\log(1+\kappa^2) - n\log\sigma - 2S(\mu,\sigma,\kappa)$$

where

$$S(\mu,\sigma,\kappa) = \sum_{i=1}^{n} logS_i(\mu,\sigma,\kappa) = \sum_{i=1}^{n} log\left[1+\frac{\kappa}{\sigma}(x_i-\mu)^+ + \frac{1}{\sigma\ \kappa}(x_i-\mu)^-\right],$$

and $(x-\mu)^+ = (x-\mu)$, if $x > \mu$, and $= 0$ otherwise, and $(x-\mu)^- = (\mu - x)$, if $x \le \mu$, and $= 0$ otherwise.

Existence, uniqueness and asymptotic normality of maximum likelihood estimators (MLEs) can be derived on the same lines as described in detail for an AL distribution in (Kotz et al. (2001)).

MLEs of σ and κ for given $\mu = \hat{\mu}$ are obtained by solving the score equations. This leads to the following equations which are solved iteratively.

$$\hat{\sigma} = \frac{2}{n} \sum_{i=1}^{n} \frac{\hat{\kappa}(x_i - \hat{\mu})^+ + \frac{1}{\hat{\kappa}}(x_i - \hat{\mu})^-}{S_i}$$

.

$$\hat{\kappa^2} = \frac{\sum_{i=1}^{n}(x_i - \hat{\mu})^-/S_i}{\sum_{i=1}^{n}(x_i - \hat{\mu})^+ S_i},$$

were $S_i(\mu, \sigma, \kappa) = \left[1 + \frac{\kappa}{\sigma}(x_i - \mu)^+ + \frac{1}{\sigma\ \kappa}(x_i - \mu)^-\right]$.

In our illustrations, the maximisation of the likelihood is implemented using the `optim` function of the R statistical software, applying the BFGS algorithm of `optim` function (Nash (1990)) in R (R Core Team (2020)). Estimates of the standard errors were obtained by inverting the numerically differentiated information matrix at the maximum likelihood estimates. We discuss the performance of our numerical maximisation algorithm using the simulated data sets in the next section.

5 Simulation

In this section we provide the simulation studies for various choices of parameters to evaluate the performance of the estimation procedure. We generated 1000 samples, each of size $n = 50, 100$ from the *ADL* distribution for $\mu = 0.5$, $\sigma = (0.5, 1)$ and $\kappa = (0.5, 1, 2)$ and then applied the algorithm to obtain the MLEs of the parameters. The results from 1000 replications are presented in Table 1. It is clear from Table 1 that the estimation algorithm works satisfactorily for various choices of parameters

Table 1: Maximum likelihood estimates of (μ, σ, κ) for various choices of parameters and n. SE stands for the asymptotic standard errors of the maximum likelihood estimates and SD is the sample standard deviations.

n	μ	σ	κ	$\hat{\mu}$	$\hat{\sigma}$	$\hat{k}$	SE($\hat{\mu}$)	SD($\hat{\mu}$)	SE($\hat{\sigma}$)	SD($\hat{\sigma}$)	SE($\hat{\kappa}$)	SD($\hat{\kappa}$)
50	0.5	0.5	0.5	0.53	0.49	0.55	0.02	0.02	0.03	0.04	0.05	0.06
		1		0.47	0.99	0.55	0.01	0.02	0.06	0.03	0.04	0.07
		0.5	1	0.54	0.49	1.07	0.01	0.01	0.04	0.05	0.05	0.08
		1		0.44	0.98	1.08	0.02	0.02	0.06	0.06	0.08	0.08
		0.5	2	0.28	0.51	2.05	0.01	0.01	0.04	0.04	0.08	0.09
		1		-0.01	1.01	1.94	0.02	0.02	0.07	0.08	0.09	0.09
100	0.5	0.5	0.5	0.48	0.49	0.49	0.01	0.01	0.02	0.02	0.02	0.02
		1		0.51	0.99	0.51	0.01	0.01	0.03	0.03	0.02	0.02
		0.5	1	0.49	0.49	1.01	0.01	0.01	0.02	0.02	0.01	0.02
		1		0.52	0.99	1.01	0.01	0.02	0.03	0.03	0.02	0.03
		0.5	2	0.47	0.51	2.01	0.01	0.01	0.02	0.02	0.02	0.03
		1		0.45	1.01	2.01	0.01	0.01	0.03	0.04	0.04	0.05

and the asymptotic standard errors of the maximum likelihood estimators agree well with the sample standard deviations over the replications.

We also checked our algorithm for various choices of initial values of parameters and sample size n. We also simulated data for varying parameters, especially towards boundary regions. When $(\sigma,\ \kappa)$ were all sufficiently away from zero, a fast convergence was observed even for arbitrary initial values. When κ and σ were very small, comparatively larger number of iterations were needed to achieve reasonable convergence.

6 Applications

In this section, we considered four real data sets to illustrate the flexibility of our proposed distribution. First one is of microarray gene expression dataset, whereas the other three are financial datasets. We compared the performance of ADL with asymmetric Laplace (AL), double Lomax (DL) and Gaussian distributions. We used Akaike's Information Criterion (AIC) (Akaike (1973) ; Burnham and Anderson (1998)), Corrected Akaike information criterion (AICC)(Hurvich (1989)), Consistent

Akaike information criterion (CAIC) (Bozdogan (1987)), Hannan-Quinn information criterion (HQIC) (Hannan and Quinn (1979)), and Bayesian Information Criterion (BIC) (Schwarz (1978)) to assess the appropriateness of ADL over the AL, DL and Gaussian distributions. The AIC, $AICC$, $CAIC$, $HQIC$ and BIC are respectively given by,

$$\begin{aligned} AIC &= -2logL + 2K, \\ AICC &= -2logL + \frac{2Kn}{n-K-1}, \\ CAIC &= -2logL + K(\log(n)+1), \\ HQIC &= -2logL + K\log(\log(n)), \\ BIC &= -2logL + K\log(n). \end{aligned}$$

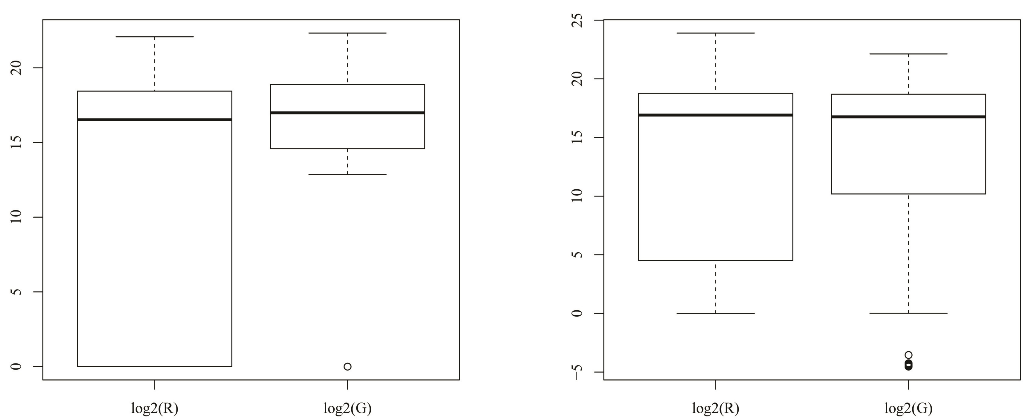
where $logL = \log(L_f(\hat{\Theta}|x_1, \cdots, x_n))$ is the log-likelihood of the data $x_1, \cdots, x_n$ under the probability distribution f, K is the number of parameters being estimated, $\hat{\Theta}$ is the maximum likelihood estimate of the parameters of f and n is the sample size. In most cases all of the above are of similar nature and give consistent results for model selection. The best model is the one which yield smaller values for these statistic and are considered to provide better fit to the data.

6.1 Microarray Gene Expression Data

In this section we apply the ADL distribution to model the distribution of a cDNA dual dye NT_2_3 microarray gene expression dataset. The microarray gene expression intensities were normalized using (Lowess) locally weighted linear regression method (Cleveland and Delvin (1988)). This method is capable of removing intensity dependence in $log_2(R_i/G_i)$ values and it has been successfully applied to microarray data (Yang et al. (2002)), where R_i is the red dye intensity and G_i is the green dye intensity for the i^{th} gene. Figure 6 is the box plots of intensities before and after Lowess normalization. The descriptive statistics for NT_2_3 microarray dataset with 16972 observations is given below in Table 2 and from the Table 2 we can see that NT_2_3 microarray dataset is positively skewed and highly peaked.

Table 2: Descriptive statistics for NT_2_3 microarray dataset

Min	Max	Mean	Median	SD	Skewness	Kurtosis
-2.865	2.564	0.011	-0.025	0.283	0.188	15.55

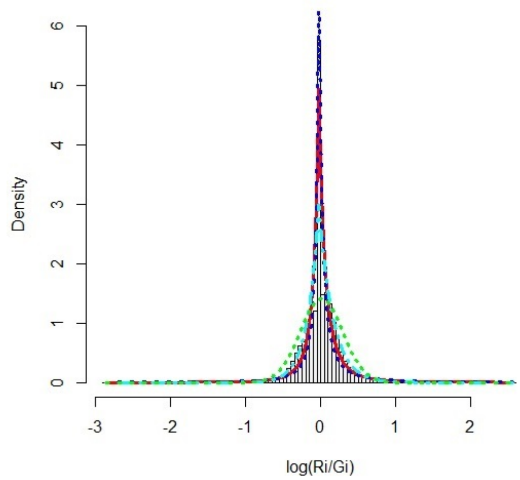


(a) (b)

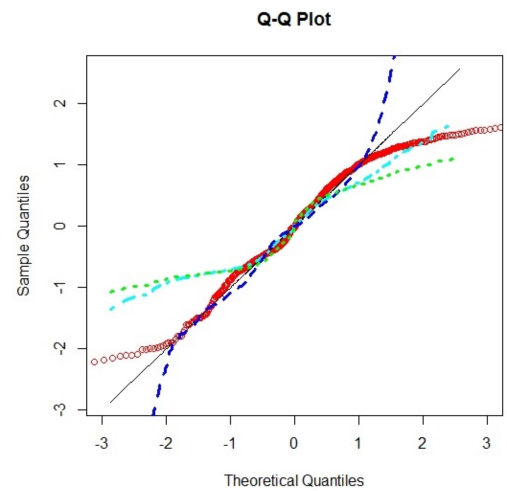
Figure 6: Box plots of intensities from microarray Experiment NT_2_3 (a) Before normalization, (b) After loess normalization.

We fitted the asymmetric double Lomax (ADL), asymmetric Laplace (AL) and Gaussian distributions to the normalized microarray intensities. The maximum likelihood estimates of the parameters and standard errors (SE) are reported in Table 3. We also plotted the quantile-quantile (Q-Q) plots since they better emphasizes the fit of the distribution in the tails.

Here we fitted Gaussian, AL, DL and ADL distributions to log-transformed normalised intensities $log_2(R_i/G_i)$ from the microarray dataset. We obtained maximum likelihood estimates and their asymptotic standard errors for the parameters of $ADL(\mu, \sigma, \kappa)$, $AL(\mu, \sigma, \kappa)$ and Gaussian $N(\mu, \sigma^2)$ distributions for the array (see Table 3 and Figure 7). Left panel of Figure 7 depicts a histogram of the gene expression data, the fitted ADL, AL, DL and Gaussian density functions evaluated at the



(a)



(b)

Figure 7: (a) Fitted ADL (red line), AL (cyan dash dot line), DL (blue dash line) and Gaussian (green dotted line), (b) Q-Q plot of ADL (red circle), AL (cyan dash dotted line), DL (blue dash line) and Guassian (green dotted line) to NT_2_3.

Table 3: Application - maximum likelihood estimates and their standard errors for ADL, AL, DL and Gaussian distributions for NT_2_3 microarray dataset are given in the table below.

	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\kappa}$	AIC	$AICC$	$CAIC$	$HQIC$	BIC
ADL	-0.025 (0.007)	0.225 (0.003)	0.878 (0.005)	3402	3402	3428	3403	3425
AL	-0.024 (0.003)	0.104 (0.001)	0.889 (0.004)	5218	5218	5258	5242	5256
DL	-0.025 (0.009)	0.066 (0.003)	-	3640	3641	3658	3641	3656
Guassian	0.011 (0.002)	0.283 (0.001)	-	6003	6004	6021	6004	6019

MLEs and right panel of Figure 7 depicts a Q-Q plot of the gene expression data, the fitted ADL, AL, DL and Guassian density functions evaluated at the MLEs. Notice, the parameter κ is a comparable measure of skewness across the datasets regardless of the scale. From figure 7 we can see that the $ADL(\mu, \sigma, \kappa)$ distribution captures three main features of the microarray data; skewness, peakedness and heavier tails. It can be seen in the right panel of Figure 7 that the fitted quantiles for ADL and DL fall close to the sample quantiles compared to that of AL and Gaussian. This clearly shows an improvement in the model fit over DL, AL and Gaussian. Only a few points out of the thousands of observations deviate significantly from a straight line and the fit of the distributions would be improved if extreme observations are removed. This indicates a good fit to the ADL and some improvement over DL. DL does not captures the skewness but it account the peakedness and heavier tail of the data.

From Table 3 we can see that the AIC, $AICC$, $CAIC$, $HQIC$ and BIC for the $ADL(\mu, \sigma, \kappa)$ had a lower value compared to DL, AL and Gaussian distributions. A smaller values indicates a better fit, and hence, ADL seems to fit the data better than DL, AL or Gaussian distributions.

6.2 Currency Exchange Rate Data

In this section we apply the ADL distribution to model the distribution of daily US dollar-Indian rupee exchange rate data from 1^{st} January 2010 to 31^{st} December 2019. Data is collected from the official website of Reserve Bank of India and https://www.exchangerates.org.uk/. The basic assumption of financial modelling has been the normality assumption for $log(X_{(t+1)}/X_t)$. Usually currency exchange rate data exhibits heavier tails, peakedness and asymmetry. To illustrate the application of ADL for this financial data, the histogram of the transformed data ($log(X_{(t+1)}/X_t)$) is plotted along with pdfs of distributions of ADL, AL, DL and Gaussian. The descriptive statistics and maximum likelihood estimates of the parameters with standard errors (SE) are presented in Table 4 and Table 5 respectively. We also plotted the quantile-quantile (Q-Q) plots since they better emphasizes the

Table 4: Descriptive statistics for US dollar to Indian rupee exchange rate ($log(X_{(t+1)}/X_t)$) dataset

Min	Max	Mean	Median	SD	Skewness	Kurtosis
-0.031	0.040	0.0002	0.001	0.005	0.220	8.616

Table 5: Maximum likelihood estimates and their standard errors for ADL, AL, DL and Gaussian distributions for US dollar to Indian rupee exchange rate ($log(X_{(t+1)}/X_t)$) dataset are given in the table below.

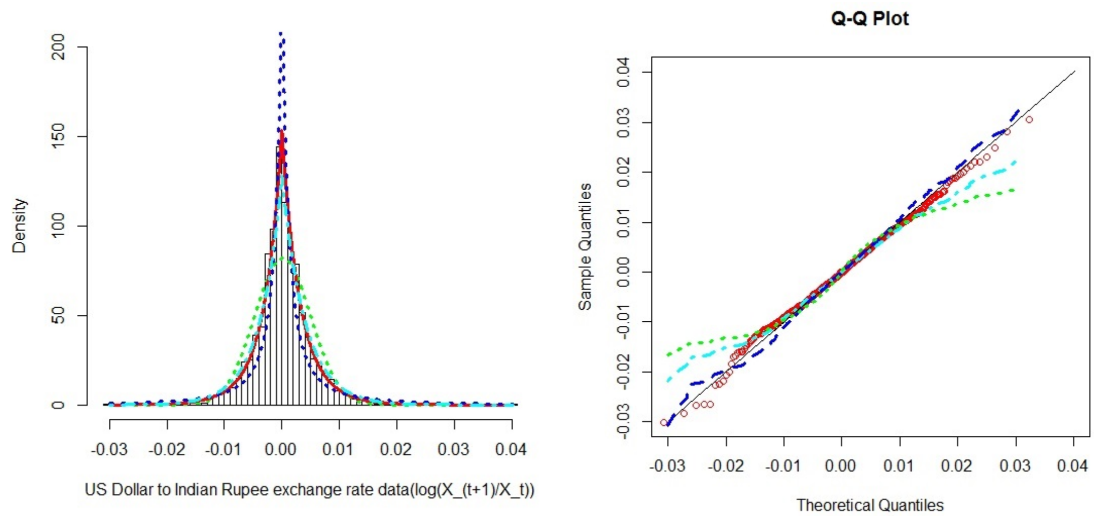
	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\kappa}$	AIC	$AICC$	$CAIC$	$HQIC$	BIC
ADL	0.001 (0.001)	0.041 (0.013)	0.977 (0.022)	9023	9024	9044	9024	9041
AL	0.001 (0.003)	0.004 (0.001)	0.963 (0.021)	9165	9165	9186	9165	9183
DL	0.001 (0.001)	0.098(0.023)	-	9664	9665	9678	9665	9676
Guassian	0.001 (0.001)	0.005 (0.001)	-	9942	9943	9956	9943	9954

fit of the distribution in the tails. We used AIC, $AICC$, $CAIC$, $HQIC$ and BIC to assess the appropriateness of ADL over the AL, DL and Gaussian distributions.

Figure 8 depicts a histogram of the currency exchange rate data, the fitted ADL, AL , DL and Gaussian density functions evaluated at the MLEs and from Table 5 we can see that the AIC, $AICC$, $CAIC$, $HQIC$ and BIC for the $ADL(\mu, \sigma, \kappa)$ had a lower value compared to DL, AL and Gaussian distributions. Smaller values for these information criteria's indicates better fit, and hence, ADL fit the data better than DL, AL or Gaussian distributions.

6.3 SENSEX Data Analysis

In this subsection we apply the ADL to model the distribution of daily S & P SENSEX data of Bombay stock exchange for the period 1 Jan 2010 to 31 December 2019, collected from the website of BSE, India. The descriptive statistics and maximum likelihood estimates of the parameters with standard errors (SE) are reported in Table 6 and Table 7.

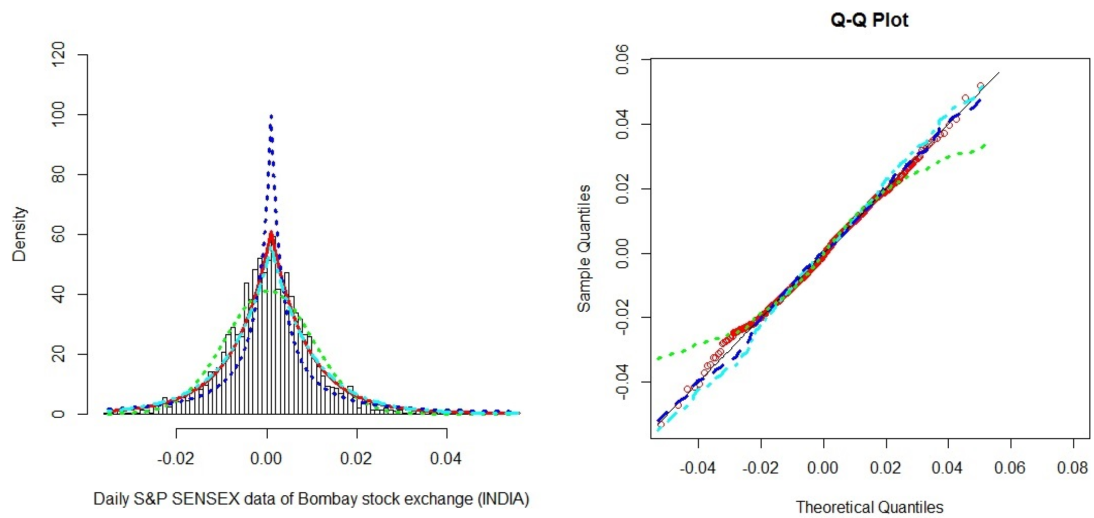


(a) (b)

Figure 8: (a) Fitted ADL probability density function (red line), AL density function(cyan dash dot line), DL (blue dash line) and Gaussian density function (green dotted line) to the US dollar to Indian rupee exchange rate ($log(X_{(t+1)}/X_t)$)(left panel),(b) Q-Q plot of ADL (red circle), AL (cyan dash dotted line), DL (blue dash line) and Gaussian (green dotted line) density functions (right panel).

Table 6: Descriptive statistics for S & P SENSEX dataset

Min	Max	Mean	Median	SD	Skewness	Kurtosis
-0.053	0.067	0.0003	0.001	0.011	0.044	5.472



(a) (b)

Figure 9: (a) Fitted ADL probability density function (red line), AL density function(cyan dash dot line), DL (blue dash line) and Gaussian density function (green dotted line) to the S & P SENSEX data ($log(X_{(t+1)}/X_t)$) (left panel),(b) Q-Q plot of ADL (red circle), AL (cyan dash dotted line), DL (blue dash line) and Gaussian (green dotted line) density functions (right panel).

From Figure 9 we observe a best fit for ADL and from Table 7 we can see that the AIC, $AICC$, $CAIC$, $HQIC$ and BIC for the $ADL(\mu, \sigma, \kappa)$ had a lower value compared to DL, AL and Gaussian distributions. Smaller values indicates a better fit, and hence, ADL fit the data better than DL, AL or Gaussian distributions.

6.4 Gold Price Data Analysis

In this section we illustrate the application of ADL distribution to model the distribution of weekly gold price per gram for the period, 30 December 2009 to 3 January

Table 7: Maximum likelihood estimates and their standard errors for ADL, AL, DL and Gaussian distributions for S & P SENSEX dataset are given in the table below

	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\kappa}$	AIC	$AICC$	$CAIC$	$HQIC$	BIC
ADL	0.001 (0.001)	0.113 (0.003)	1.008 (0.005)	5053	5054	5074	5071	5071
AL	0.001 (0.001)	0.008 (0.001)	1.017 (0.024)	5871	5872	5892	5872	5589
DL	0.001 (0.001)	0.006 (0.001)	-	5871	5872	5886	5872	5884
Guassian	0.001 (0.001)	0.011 (0.001)	-	6942	6943	6956	6943	6954

Table 8: Descriptive statistics for weekly gold price (per gram) data ($log(X_{(t+1)}/X_t)$)

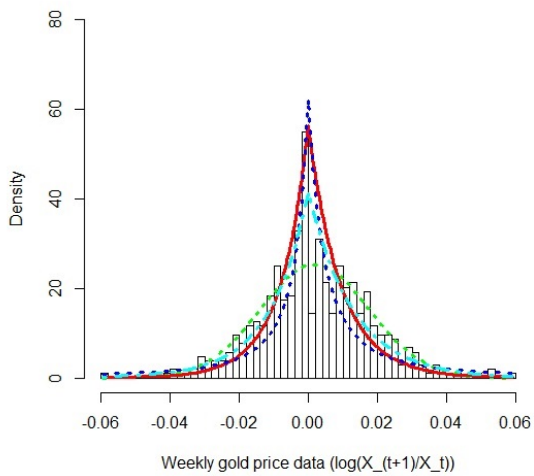
Min	Max	Mean	Median	SD	Skewness	Kurtosis
-0.098	0.071	0.002	0	0.017	-0.377	6.859

2020, collected from the website www.keralagold.com. Descriptive statistics is given in Table 8 and maximum likelihood estimates of the parameters and standard errors (SE) are reported in Table 9.

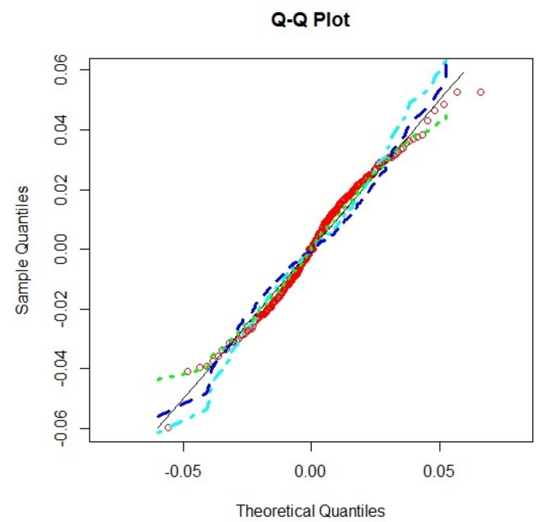
From Figure 10 we observe the goodness of fit for ADL, compared to AL, DL and Gaussian density functions and is confirmed by the lower values for the AIC, $AICC$, $CAIC$, $HQIC$ and BIC for the $ADL(\mu, \sigma, \kappa)$ compared to DL, AL and Gaussian distributions.

Table 9: Maximum likelihood estimates and their standard errors for ADL, AL, DL and Gaussian distributions for weekly gold price (per gram) dataset ($log(X_{(t+1)}/X_t)$) are given in the table below

	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\kappa}$	AIC	$AICC$	$CAIC$	$HQIC$	BIC
ADL	0.001 (0.001)	0.143 (0.003)	0.935(0.021)	1154	1154	1170	1153	1167
AL	0.001 (0.001)	0.013 (0.001)	0.939 (0.034)	1266	1266	1282	1266	1279
DL	0.001 (0.001)	0.008(0.001)	-	1230	1231	1241	1231	1239
Guassian	0.002 (0.002)	0.018 (0.001)	-	1280	1281	1291	1290	1299



(a)



(b)

Figure 10: (a) Fitted ADL probability density function (red line), AL density function(cyan dash dot line), DL (blue dash line) and Gaussian density function (green dotted line) to the weekly gold price (per gram) data $(log(X_{(t+1)}/X_t))$(left panel), (b) Q-Q plot of ADL (red circle), AL (cyan dash dotted line), DL (blue dash line) and Gaussian (green dotted line) density functions (right panel).

7 Conclusion

Punathumparambath and Kulathinal (2015) introduced the double Lomax distribution (DL), which is a flexible distribution that can adjust for heavy tails. Cauchy distribution which is the ratio of i.i.d standard normal is log symmetric and heavy tailed, in the same way the double Lomax random variable is the ratio of two i.i.d. classical Laplace variables is log symmetric and heavy tailed. We created the asymmetric double Lomax (ADL) distribution since DL is not ideal for modelling asymmetric data. ADL is beneficial in analysing datasets that are asymmetric, leptokurtic, and having heavy tail structure, these are some of the common characteristics of datasets in financial modelling and microarray modelling.

We used the proposed probability distribution, a generalisation of the asymmetric Laplace distribution, to analyze differential gene expression and financial data in this article. Because the distribution of normalised gene expressions, while similar across arrays, is frequently far from normal, the distribution reveals larger tails and asymmetry of variable degrees even after normalisation. The model proposed in this article can be used to simulate the impulsiveness and skewness that are required for gene expression data. Our distribution is a proper distribution to accept outliers in the data since it is heavy-tailed. When it comes to microarray gene expression data, the number of genes that are differentially expressed is typically a small percentage of the total number of genes analysed. As a result, the probability distribution given in this research will be highly useful in situations involving gene expression data estimation and detection.

For financial modelling, the data on currency exchange rate, BSE SENSEX and gold price is observed to have better fit with ADL than AL, DL and Gaussian distributions. ADL provides a lot of versatility in terms of tail behaviour and peakedness, and it may be used on data from a variety of sources. The ADL gives the Laplace distribution flexibility in terms of skewness and tail behaviour. The Laplace distribution, as well as its various skewed and heavy-tailed generalisations, offers a viable alternative to Gaussian-based stochastic models.

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