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Bivariate Generalized Discrete Modified Weibull Distribution

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ABSTRACT

In this paper, we introduce a new generalization of univariate and bivariate modified discrete Weibull distribution. Various properties of univariate generalized modified discrete Weibull distribution such as survival function, probability mass function, hazard rate function, probability generating function, moment generating function are derived. The joint distribution function, joint probability mass function, marginal distributions, moment generating function, conditional distribution of proposed bivariate distribution are derived. Parameters of the distributions are estimated using Maximum likelihood estimation. The use of these distributions are illustrated using real life data sets.

Key words and Phrases: *Bivariate distribution, Discrete distribution Generalized modified Weibull distribution, Maximum likelihood estimation.*

1 Introduction

There are many fields in which researchers regularly encounter variables that are discrete in nature as part of their research. Therefore, continuous models may be unsuitable in such cases and the observed values can be recorded in a way that makes discrete models more appropriate in such cases. A number of continuous

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lifetime distributions have been derived over the last few decades, and some of them have been studied and modified to suit various applications.There are comparatively fewer studies on discrete distributions than there are on continuous distributions. So, there is a need for more studies in this field.

In recent years, many researchers have focused their attention on the problem of constructing discrete analogues of continuous probability distributions. Chakraborty (2015) provides discrete analogues to many distributions that are known to exist. It has been proposed by Nakagawa and Osaki (1975), Stein and Dattero (1984) as well as Padgett and Spurrier (1985) that there are three types of discrete Weibull distributions. A two-parameter discrete gamma distribution was introduced by Yang (1994). Roy (2003) proposed the discrete normal distribution. Roy (2004) introduced the discrete Rayleigh distribution. The discrete Burr distribution and the discrete Pareto distribution were proposed by Krishna and Singh (2009). Jazi et al (2010) proposed a discrete analogue the inverse Weibull distribution. A discrete version of the continuous modified Weibull distribution (Lai et al, 2003) was introduced by Noughabi et al (2011). Bebbington et al (2012) proposed a discrete analogue of additive Weibull. Al-Huniti and Al-Dayian (2012) proposed discrete Burr type III. Bakouch et al (2012) introduced discrete Lindley distribution. A discrete Weibull geometric distribution is proposed by Jayakumar and Babu (2017). Also Jayakumar and Babu (2019) introduced discrete additive Weibull geometric distribution.

Over a long period of time, bivariate distributions have attracted the attention of a number of researchers. In many fields, it is important to be able to model data based on these distributions. There are many real life situations in which discrete bivariate data arises that are often highly correlated with one another. There are various bivariate discrete distributions in the literature that can be studied. Kocherlakota and Kocherlakota (1992) discussed about many different discrete bivariate distributions. It is quite natural for discrete bivariate data to emerge in many different situations in real life. Based on the minimization and maximization methods Lee and Cha (2015) proposed two general classes of discrete bivariate distributions. A four-parameter bivariate discrete generalized exponential distribution proposed

by Nekoukhou and Kundu (2017). Kundu and Nekoukhou (2018a) proposed an univariate and a bivariate geometric discrete generalized exponential distributions. Kundu and Nekoukhou (2018b) in their paper, discussed about a new bivariate discrete Weibull distribution which can be taken as a discrete analogue of Marshall and Olkin bivariate Weibull distribution (see, Kundu and Gupta, 2013). El-Morshedy et al (2020) introduced bivariate discrete exponentiated Weibull distribution. Eliwa and El-Morshedy (2018) proposed a bivariate discrete inverse Weibull distribution. Shibu and Beegum (2021) introduced a bivariate discrete modified Weibull distribution, which is a discrete analogue of the new bivariate modified Weibull distribution (see, El-Bassiouny et al, 2018).

Here we use a new method to find the joint survival function of the both univariate and bivariate discrete modified Weibull distribution. The rest of this study is arranged as follows. Section 2 discusses some preliminaries. Section 3 introduces the bivariate extension of the discrete modified Weibull distribution and its properties. Section 4 contains the maximum likelihood estimation of the bivariate distribution and section 5 involves data analysis. Section 6 devotes to simulation study of the bivariate generalized discrete modified Weibull distribution, followed by conclusion in Section 7.

2 Preliminaries

Consider a sequence of Bernoulli trials in which the i-th trial has probability of success $\frac{\nu}{i}$, $0 < \nu \leq 1$, $i \in \{1, 2, \dots\}$. Let the trails are independent. Let K denote the trial number on which the first success occurs. Then the probability mass function and probability generating function of random variable K is given by

$$\begin{aligned} P(K = k) &= (1-\nu)\left(1 - \frac{\nu}{2}\right) \dots \left(1 - \frac{\nu}{k-1}\right)\frac{\nu}{k} \\ &= \frac{(-1)^{k-1}}{k!}\nu(\nu-1)\dots(\nu-k+1) \end{aligned} \tag{2.1}$$

for $k = 1, 2, \dots$ and

$$\begin{aligned} h_K(s) &= E(s^K) \\ &= 1 - (1-s)^\nu, s \in [0,1], \end{aligned}$$

respectively. (see, Dolati et. al, 2014, Mirhosseini et. al 2015, Pathak and Vellaisamy, 2020)

2.1 Univariate Generalized Discrete Modified Weibull Distribution

Consider that $\{U_1, U_2, \dots\}$ a sequence of mutually independent and identically distributed random variables, where $U_i \sim DMW(q_1, \theta, c)$ for $i \in \{1, 2, \dots\}$. Define $X = \min(U_1, U_2, \dots, U_k)$. Then X is said to follow univariate generalized discrete modified Weibull distribution with parameters q_1, θ, c and ν, i.e., $X \sim GDMW(q_1, \theta, c, \nu)$. Using equation (2.1), we can derive the survival function of $GDMW(q_1, \theta, c, \nu)$ as

$$\begin{aligned} S_{GDMW}(x) &= 1 - (1 - q_1^{x^\theta c^x})^\nu \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\nu(\nu-1)\dots(\nu-(n-1))}{n!} q_1^{n x^\theta c^x}. \end{aligned}$$

3 Bivariate Generalized Discrete Modified Weibull Distribution

Consider that $\{U_1, U_2, \dots\}$ and $\{V_1, V_2, \dots\}$ are two sequences of mutually independent and identically distributed random variables, where $U_i \sim DMW(q_1, \theta, c)$ and $V_i \sim DMW(q_2, \theta, c)$ for $i \in \{1, 2, \dots\}$. Define $X = \min(U_1, U_2, \dots, U_k)$ and $Y = \min(V_1, V_2, \dots, V_k)$. Then (X, Y) is said to follow bivariate generalized discrete modified Weibull distribution with parameters q_1, q_2, θ, c and ν, $(X, Y) \sim BGDMW(q_1, q_2, \theta, c, \nu)$.

3.1 Joint Survival Function

The joint survival function of bivariate generalized discrete modified Weibull distribution with parameters q_1, q_2, θ, c and ν is given by

$$S(x,y) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\nu(\nu-1)\dots(\nu-(n-1))}{n!} q_1^{nx^\theta c^x} q_2^{ny^\theta c^y}.$$

Proof. Let $(X,Y) \sim BGDMW(q_1, q_2, \theta, c, \nu)$, then the joint survival function can be derived as,

$$\begin{aligned}
S(x,y) &= P(X > x, Y > y) \\
&= P(\min(U_1, U_2, \dots, U_k) > x, \min(V_1, V_2, \dots, V_k) > y) \\
&= \sum_{k=1}^{\infty} (P(U_i > x)P(V_i > y))^k P(K = k) \\
&= h_K(q_1^{x^\theta c^x} q_2^{y^\theta c^y}) \\
&= 1 - (1 - q_1^{x^\theta c^x} q_2^{y^\theta c^y})^\nu \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\nu(\nu-1)\dots(\nu-(n-1))}{n!} q_1^{nx^\theta c^x} q_2^{ny^\theta c^y}.
\end{aligned}$$

Hence the proof. □

The joint survival function is plotted for two different sets of values of the parameters.

3.2 Joint Probability Mass Function

If $(X,Y) \sim BGDMW(q_1, q_2, \theta, c, \nu)$, then joint probability mass function of (X,Y) is given by

$$f(x,y) = \sum_{n=1}^{\infty} \sum_{i=0}^{1} \sum_{j=0}^{1} \frac{(-1)^{n+i+j+2}}{n!} (\nu(\nu-1)\dots(\nu-(n-1))) q_1^{n(x+i)^\theta c^{x+i}} q_2^{n(y+j)^\theta c^{y+j}}.$$

Proof. Consider a bivariate generalized discrete modified Weibull distribution with parameters q_1, q_2, θ, c and ν.
We have

$$f(x,y) = S(x,y) - S(x,y+1) - S(x+1,y) + S(x+1,y+1).$$

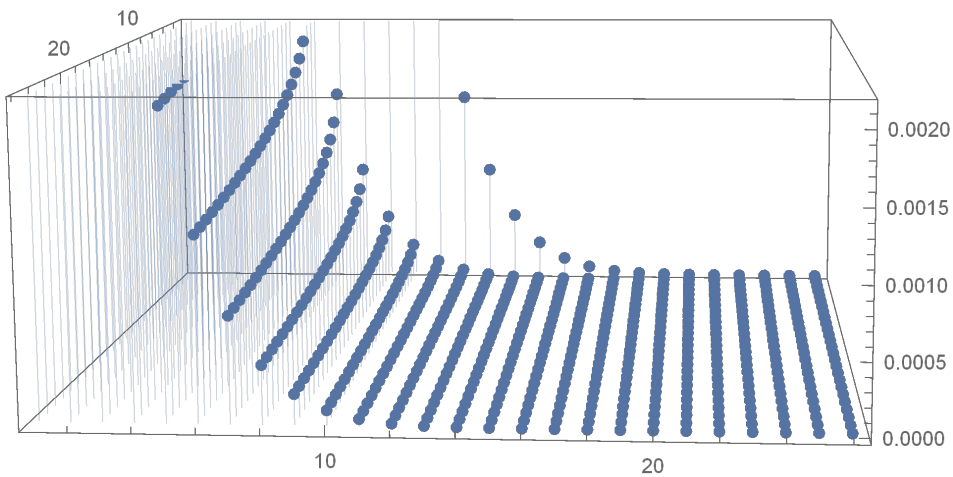

Figure 1: The joint SF of a BGDMW distribution with parameters $q_1 = 0.2, q_2 = 0.3, \theta = 0.1, c = 1.09, \nu = 0.3$.

Thus, the joint probability mass function of (X, Y) is derived as

$$\begin{aligned} f(x,y) &= (1 - q_1^{x^\theta c^x} q_2^{(y+1)^\theta c^{(y+1)}})^\nu + (1 - q_1^{(x+1)^\theta c^{(x+1)}} q_2^{y^\theta c^y})^\nu \\ &\quad - (1 - q_1^{x^\theta c^x} q_2^{y^\theta c^y})^\nu - (1 - q_1^{(x+1)^\theta c^{(x+1)}} q_2^{(y+1)^\theta c^{(y+1)}})^\nu \\ &= \sum_{n=1}^{\infty} \sum_{i=0}^{1} \sum_{j=0}^{1} \frac{(-1)^{n+i+j+2}}{n!} (\nu(\nu-1)\dots(\nu-(n-1))) q_1^{n(x+i)^\theta c^{x+i}} q_2^{n(y+j)^\theta c^{y+j}}. \end{aligned}$$

□

In figure 3 and 4 the joint probability mass function (PMF) is plotted for two different sets of values of the parameters.

3.3 Joint Distribution Function

Let $(X, Y) \sim BGDMW(q_1, q_2, \theta, c, \nu)$, then joint distribution function of (X, Y) is

$$F(x,y) = 3 - \sum_{n=1}^{\infty} \sum_{\substack{i=0 \\ i=j\neq 0}}^{1} \sum_{j=0}^{1} \frac{(-1)^{n+1}}{n!} (\nu(\nu-1)\dots(\nu-(n-1))) q_1^{ni^\theta c^i} q_2^{nj^\theta c^j}.$$

Proof. The distribution function of a bivariate generalized discrete modified Weibull

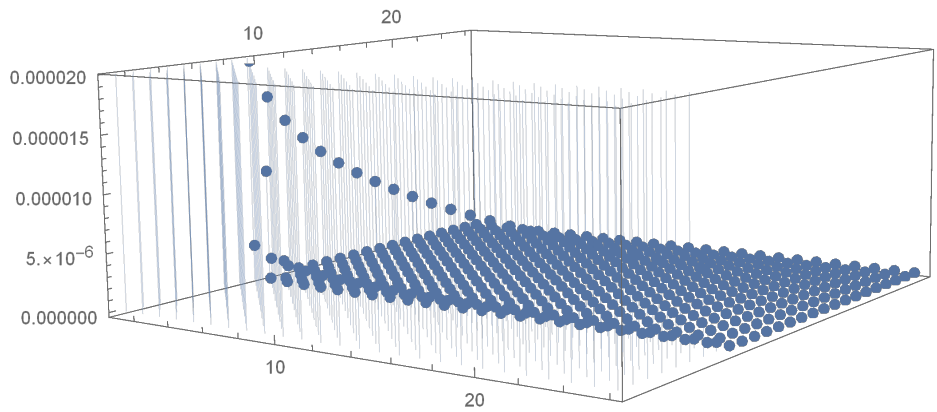

Figure 2: The joint SF of a BGDMW distribution with parameters $q_1 = 0.2, q_2 = 0.3, \theta = 0.1, c = 1.2, \nu = 0.3$.

distribution with parameters q_1, q_2, θ, c and ν can be derived as

$$\begin{aligned} F(x,y) &= F_1(x) + F_2(y) + S(x,y) - 1 \\ &= (1 - q_1^{x^\theta c^x})^\nu + (1 - q_2^{y^\theta c^y})^\nu + (1 - q_1^{x^\theta c^x} q_2^{y^\theta c^y})^\nu \\ &= 3 - \sum_{n=1}^{\infty} \sum_{\substack{i=0 \\ i=j\neq 0}}^{1} \sum_{j=0}^{1} \frac{(-1)^{n+1}}{n!} (\nu(\nu-1)\dots(\nu-(n-1)))q_1^{ni^\theta c^i} q_2^{nj^\theta c^j}. \end{aligned}$$

□

3.4 Conditional Survival Function of X given $Y > y$

If $(X,Y) \sim BGDMW(q_1, q_2, \theta, c, \nu)$, then the conditional survival function of X given $Y > y$ is given by

$$S_{X/Y>y}(x) = \frac{1 - (1 - q_1^{x^\theta c^x} q_2^{y^\theta c^y})^\nu}{1 - (1 - q_2^{y^\theta c^y})^\nu}.$$

Proof. Let (X,Y) be bivariate generalized discrete modified Weibull distribution with parameters q_1, q_2, θ, c and ν, then the conditional survival function of X given

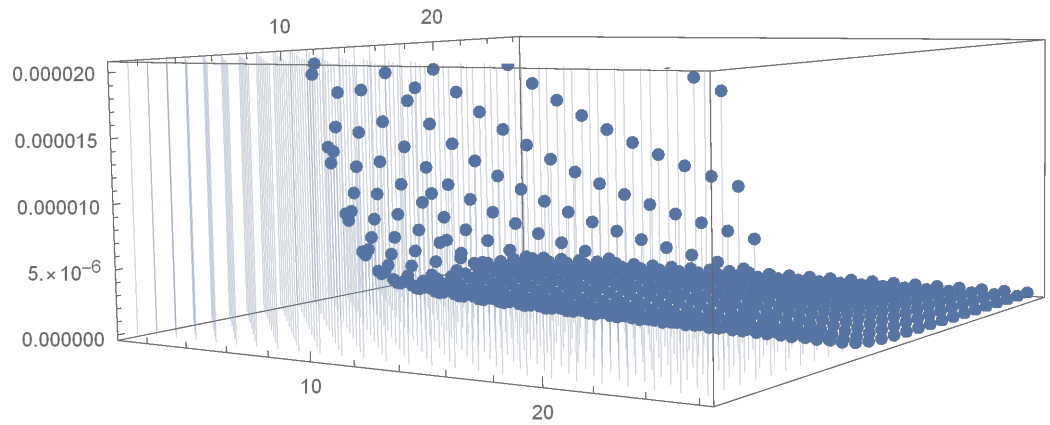

Figure 3: The joint PMF of a BGDMW distribution with parameters $q_1 = 0.2, q_2 = 0.3, \theta = 0.1, c = 1.09, \nu = 0.3$.

$Y > y$ can be obtained as

$$\begin{aligned} S_{X/Y>y}(x) &= P(X > x/Y > y) \\ &= \frac{P(X > x, Y > y)}{P(Y > y)} \\ &= \frac{1 - (1 - q_1^{x^\theta c^x} q_2^{y^\theta c^y})^\nu}{1 - (1 - q_2^{y^\theta c^y})^\nu}. \end{aligned}$$

□

3.5 Conditional survival function of X given $Y > y$

If (X, Y) is bivariate generalized discrete modified Weibull distribution with parameters q_1, q_2, θ, c and ν.The conditional distribution function of X given $Y \leq y$ is

$$F_{X/Y\leq y}(x) = 1 + \frac{(1 - q_1^{x^\theta c^x})^\nu}{(1 - q_2^{y^\theta c^y})^\nu} + \frac{(1 - q_1^{x^\theta c^x} q_2^{y^\theta c^y})^\nu}{(1 - q_2^{y^\theta c^y})^\nu}$$

Proof. Let $(X, Y) \sim BGDMW(q_1, q_2, \theta, c, \nu)$, then the conditional distribution of

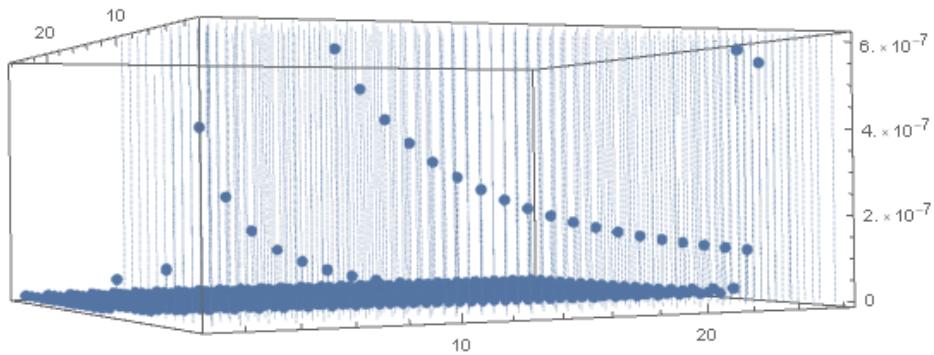

Figure 4: The joint PMF of a BGDMW distribution with parameters $q_1 = 0.2, q_2 = 0.3, \theta = 0.1, c = 1.2, \nu = 0.3$.

X given $Y \leq y$ is

$$\begin{aligned} F_{X/Y\leq y}(x) &= P(X \leq x/Y \leq y) \\ &= \frac{P(X \leq x, Y \leq y)}{P(Y \leq y)} \\ &= \frac{(1-q_1^{x^\theta c^x} q_2^{y^\theta c^y})^\nu + (1-q_1^{x^\theta c^x})^\nu + (1-q_2^{y^\theta c^y})^\nu}{(1-q_2^{y^\theta c^y})^\nu} \\ &= 1 + \frac{(1-q_1^{x^\theta c^x})^\nu}{(1-q_2^{y^\theta c^y})^\nu} + \frac{(1-q_1^{x^\theta c^x} q_2^{y^\theta c^y})^\nu}{(1-q_2^{y^\theta c^y})^\nu}. \end{aligned}$$

□

3.6 Conditional PMF of X given $Y = y$

The conditional pmf of X given $Y = y$ is given by

$$f_{X/Y=y}(x) = \frac{\sum_{n=1}^{\infty}\sum_{i=0}^{1}\sum_{j=0}^{1}\frac{(-1)^{n+i+j+2}}{n!}(\nu(\nu-1)\ldots(\nu-(n-1)))q_1^{n(x+i)^\theta c^{x+i}} q_2^{n(y+j)^\theta c^{y+j}}}{\sum_{n=1}^{\infty}\sum_{j=0}^{1}(-1)^{n+j+2}\frac{\nu(\nu-1)\ldots(\nu-(n-1))}{n!}q_2^{n(y+j)^\theta c^{y+j}}}.$$

Proof. The proof is trivial. □

Joint Hazard Rate Function

If $(X,Y) \sim BGDMW(q_1,q_2,\theta,c,\nu)$, the joint hazard rate function of (X,Y) is given by

$$h(x,y) = 1 - \frac{1-(1-q_1^{x^\theta c^x} q_2^{(y+1)^\theta c^{y+1}})^\nu - (1-q_1^{(x+1)^\theta c^{x+1}} q_2^{y^\theta c^y})^\nu + (1-q_1^{(x+1)^\theta c^{x+1}} q_2^{(y+1)^\theta c^{y+1}})^\nu}{1-(1-q_1^{x^\theta c^x} q_2^{y^\theta c^y})^\nu}$$

Proof. The proof is trivial. □

The joint hazard rate function for a parameter set $q_1 = 0.2, q_2 = 0.3, \theta = 0.1, c = 1.09$ and $\nu = 0.3$ is given in figure 5.

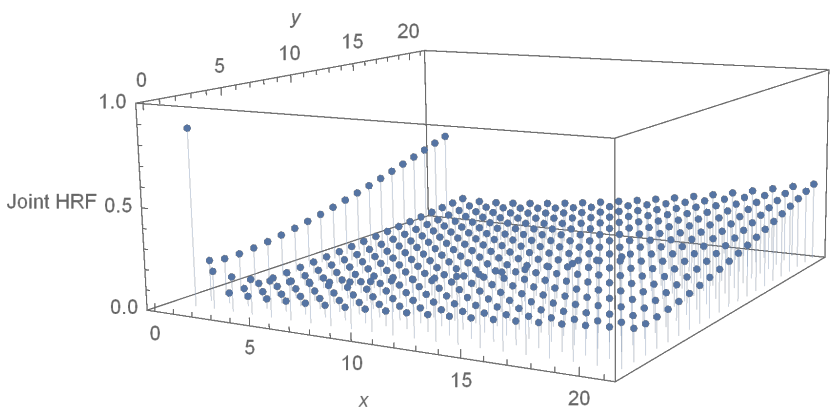

Figure 5: The joint HRF of a BGDMW distribution with parameters $q_1 = 0.2, q_2 = 0.3, \theta = 0.1, c = 1.09, \nu = 0.3$.

Theorem 3.1. *If* $(X,Y) \sim BGDMW(q_1,q_2,\theta,c,\nu)$ *then* $min(X,Y) \sim GDMW(q_1q_2,\theta,c,\nu)$.

Proof. Let (X,Y) is bivariate generalized discrete modified Weibull distribution with parameters q_1, q_2, θ, c and ν

$$\begin{aligned} P(\min(X,Y) > s) &= P(X > s, Y > s) \\ &= \sum_{k=1}^{\infty} (P(U_i > s)P(V_i > s))^k P(K = k) \\ &= h_k((q_1q_2)^{s^\theta c^s}) \\ &= 1 - (1 - (q_1q_2)^{s^\theta c^s})^\nu. \end{aligned}$$

Thus $\min(X,Y) \sim GDMW(q_1q_2,\theta,c,\nu)$. □

4 Estimation

4.1 Maximum Likelihood Estimation

To estimate unknown parameters of bivariate generalized discrete modified Weibull distribution using maximum likelihood estimation, consider a sample of size m, $\{(x_1, y_1), (x_2, y_2), ..., (x_m, y_m)\}$ from bivariate generalized discrete modified Weibull distribution with parameters q_1, q_2, θ, c and ν.
The likelihood function $l(\underline{x}; q_1, q_2, \theta, c, \nu)$ is given by

$$l(\underline{x}; q_1, q_2, \theta, c, \nu) = \prod_{i=1}^{m} f(x_i, y_i).$$

Thus the log likelihood function L,

$$\begin{aligned} L = \sum_{i=1}^{m} ln[&\big(1 - q_1^{x_i^\theta c^{x_i}} q_2^{(y_i+1)^\theta c^{(y_i+1)}}\big)^\nu + \big(1 - q_1^{(x_i+1)^\theta c^{(x_i+1)}} q_2^{y_i^\theta c^{y_i}}\big)^\nu \\ &- \big(1 - q_1^{x_i^\theta c^{x_i}} q_2^{y_i^\theta c^{y_i}}\big)^\nu - \big(1 - q_1^{(x_i+1)^\theta c^{(x_i+1)}} q_2^{(y_i+1)^\theta c^{(y_i+1)}}\big)^\nu]. \end{aligned}$$

$$\begin{aligned} \frac{\partial L}{\partial \nu} = &\sum_{i=1}^{m} \frac{g_1(x_i+1, y_i, q_1, q_2, \theta, c, \nu) + g_1(x_i, y_i+1, q_1, q_2, \theta, c, \nu)}{f(x_i, y_i)} \\ &- \sum_{i=1}^{m} \frac{g_1(x_i, y_i, q_1, q_2, \theta, c, \nu) + g_1(x_i+1, y_i+1, q_1, q_2, \theta, c, \nu)}{f(x_i, y_i)}. \end{aligned}$$

$$\begin{aligned} \frac{\partial L}{\partial \theta} = &\sum_{i=1}^{m} \frac{g_2(x_i, y_i, q_1, q_2, \theta, c, \nu) + g_2(x_i+1, y_i+1, q_1, q_2, \theta, c, \nu)}{f(x_i, y_i)} \\ &- \sum_{i=1}^{m} \frac{g_2(x_i+1, y_i, q_1, q_2, \theta, c, \nu) + g_2(x_i, y_i+1, q_1, q_2, \theta, c, \nu)}{f(x_i, y_i)}. \end{aligned}$$

$$\begin{aligned} \frac{\partial L}{\partial c} = &\sum_{i=1}^{m} \frac{g_3(x_i, y_i, q_1, q_2, \theta, c, \nu) + g_3(x_i+1, y_i+1, q_1, q_2, \theta, c, \nu)}{f(x_i, y_i)} \\ &- \sum_{i=1}^{m} \frac{g_3(x_i+1, y_i, q_1, q_2, \theta, c, \nu) + g_3(x_i, y_i+1, q_1, q_2, \theta, c, \nu)}{f(x_i, y_i)}. \end{aligned}$$

$$\frac{\partial L}{\partial q_1} = \sum_{i=1}^{m} \frac{g_4(x_i, y_i, q_1, q_2, \theta, c, \nu) + g_4(x_i+1, y_i+1, q_1, q_2, \theta, c, \nu)}{f(x_i, y_i)} - \sum_{i=1}^{m} \frac{g_4(x_i+1, y_i, q_1, q_2, \theta, c, \nu) + g_4(x_i, y_i+1, q_1, q_2, \theta, c, \nu)}{f(x_i, y_i)}$$

$$\frac{\partial L}{\partial q_2} = \sum_{i=1}^{m} \frac{g_5(x_i, y_i, q_1, q_2, \theta, c, \nu) + g_5(x_i+1, y_i+1, q_1, q_2, \theta, c, \nu)}{f(x_i, y_i)} - \sum_{i=1}^{m} \frac{g_5(x_i+1, y_i, q_1, q_2, \theta, c, \nu) + g_5(x_i, y_i+1, q_1, q_2, \theta, c, \nu)}{f(x_i, y_i)}$$

where $g_1(x, y, q_1, q_2, \theta, c, \nu) = (1 - q_1^{x^\theta c^x} q_2^{y^\theta c^y})^\nu ln(1 - q_1^{x^\theta c^x} q_2^{y^\theta c^y})$, $g_2(x, y, q_1, q_2, \theta, c, \nu) = \nu(1-q_1^{x^\theta c^x} q_2^{y^\theta c^y})^{\nu-1} q_1^{x^\theta c^x} q_2^{y^\theta c^y} (x^\theta c^x ln(x+q_1)+y^\theta c^y ln(y+q_2))$, $g_3(x, y, q_1, q_2, \theta, c, \nu) = \nu(1 - q_1^{x^\theta c^x} q_2^{y^\theta c^y})^{\nu-1} q_1^{x^\theta c^x} q_2^{y^\theta c^y} ln(q_1^{x^{\theta+1} c^{x-1}} q_2^{y^{\theta+1} c^{y-1}})$, $g_4(x, y, q_1, q_2, \theta, c, \nu) = \nu(1-q_1^{x^\theta c^x} q_2^{y^\theta c^y})^{\nu-1} x^\theta c^x q_1^{x^\theta c^x - 1} q_2^{y^\theta c^y}$ and $g_5(x, y, q_1, q_2, \theta, c, \nu) = \nu(1 - q_1^{x^\theta c^x} q_2^{y^\theta c^y})^{\nu-1} y^\theta c^y q_1^{x^\theta c^x} q_2^{y^\theta c^y - 1}$.

The maximum likelihood estimators of the parameters q_1, q_2, θ, c and ν can be obtained by solving these five non linear equations. The solution of these equations are not easy to solve. We need a numerical technique to get the maximum likelihood estimators.

5 Data Analysis

In this section we discuss maximum likelihood estimators of the parameters of BGDMW distribution using two real life data sets and compare the new bivariate discrete modified Weibull distribution with the bivariate discrete exponential distribution, the bivariate discrete Rayleigh distribution, the bivariate discrete Weibull distribution proposed by Kundu and Nekoukhou (2018b). and bivariate discrete modified Weibull distribution proposed by Shibu and Beegum (2021) using maximized Log-Likelihood (-L), Akaike information criterion (AIC), corrected Akaike

information criterion (AICc), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQC). Here the maximum likelihood estimates of parameters are obtained by limited memory quasi Newton algorithm (Byrd et al, 1995).

Data 1

The data set given in Table 1 consists of a football match scored in Italian football match (Series A) during 1996 to 2011, between ACF Fiorentina(X_1) and Juventus (X_2).

Table 1: The Score data between ACF Fiorentina(X_1) and Juventus (X_2)

Obs.	Match Date	X_1	X_2	Obs.	Match Date	X_1	X_2
1	25/10/2011	1	2	14	16/02/2002	1	2
2	17/04/2011	0	0	15	19/12/2001	1	1
3	27/11/2010	1	1	16	12/05/2001	1	3
4	06/03/2010	1	2	17	06/01/2001	3	3
5	17/10/2009	1	1	18	21/04/2000	0	1
6	24/01/2009	0	1	19	18/12/1999	1	1
7	31/08/2008	1	1	20	24/04/1999	1	2
8	02/03/2008	3	2	21	12/12/1998	1	0
9	07/10/2007	1	1	22	21/02/1998	3	0
10	09/04/2006	1	1	23	04/10/1997	1	2
11	04/12/2005	1	2	24	22/02/1997	1	1
12	09/04/2005	3	3	25	28/09/1996	0	1
13	10/11/2004	0	1	26	23/03/1996	0	1

The values for -L, AIC, AICc, BIC and HQC of new bivariate discrete modified Weibull distribution less than the values corresponding to the discrete exponential distribution, discrete Rayleigh distribution, discrete Weibull distribution and discrete modified Weibull distribution. Values are provided in Table 2. Thus we can say that the new bivariate discrete modified Weibull distribution provide better fit

Table 2: The MLEs,-L,AIC,AICc,BIC and HQIC values for the data

	MLEs	-L	AIC	AICc	BIC	HQIC
BDE	q_0=0.652,q_1=0.812,q_2=0.713	75.35	156.70	157.79	160.47	157.79
BDR	q_0=0.790,q_1=0.872,q_2=0.905	63.99	133.98	135.07	137.75	135.07
BDW	q_0=0.99, q_1=0.781,q_2=0.952, θ=4.98	293.08	594.16	596.07	599.19	595.61
BDMW	q_0=0.95, q_1=0.895,q_2=0.939, θ=0.803, c=2.00	67.51	149.01	148.01	151.3	146.83
BGDMW	q_1=0.888,q_2=0.92, $\nu = 0.687$, θ=1.213,c=2.251	65.19	140.38	143.38	146.67	142.19

than the respective distributions.

Data 2

The data represented in Table 3 is reported in Davis (2002), and it represents the eccacy of steam inhalation in the treatment of common cold symptoms (0 = no symptoms; 1 = mild symptoms; 2 = moderate symptoms; 3 = severe symptoms).

The -L, AIC, AICc, BIC and HQC of the new bivariate discrete modified Weibull distribution has lesser value than the values corresponding to the discrete exponential distribution, discrete Weibull distribution and discrete modified Weibull distribution. Values are provided in Table 4. Thus we can say that the new bivariate discrete modified Weibull distribution provide better fit than the respective distributions.

6 Simulation Study

Consider a bivariate random variable (X, Y) with joint cumulative distribution function $F_{X,Y}(x, y)$. Statistically independent standard uniform variables U_1 and U_2,

Table 3: Nasal drainage severity score.

Obs.	Day 1 (X_1)	Day 2 X_2	Obs.	Day 1 (X_1)	Day 2 (X_2)
1	1	1	16	2	1
2	0	0	17	1	1
3	1	1	18	2	2
4	1	1	19	3	1
5	0	2	20	1	1
6	2	0	21	2	1
7	2	2	22	2	2
8	1	1	23	1	1
9	3	2	24	2	2
10	2	2	25	2	0
11	1	0	26	1	1
12	2	3	27	0	1
13	1	3	28	1	1
14	2	1	29	1	1
15	2	3	30	3	3

$U = (U_1, U_2)$ are obtained from the following equations

$$u_1 = F_X(x)$$

$$u_2 = F_{Y/X=x}(y)$$

where u_1 and U_2 are samples from the U space, $F_X(x)$ is the distribution function of X and $F_{Y/X=x}(y)$ is the conditional distribution function of Y given $X = x$. The conditional distribution $F_{Y/X=x}(y)$ be obtained from the equation

$$\begin{aligned} F_{Y/X=x}(y) &= P(Y \leq y/X = x) \\ &= \frac{1}{f_X(x)} \sum_{t \leq y} f(x,t), \end{aligned}$$

where $f_X(x)$ is the marginal density of X,

$$f_X(x) = (1 - q_1^{(x+1)^\theta c^{(x+1)}})^\nu - (1 - q_1^{x^\theta c^x})^\nu$$

Table 4: The MLEs,-L,AIC,AICc,BIC and HQIC values for the data

	MLEs	-L	AIC	AICc	BIC	HQIC
BDE	q_0=0.846,q_1=0.792,q_2=0.693	88.00	182.00	182.92	186.20	183.34
BDW	q_0=0.932, q_1=0.998,q_2=0.933, θ=3.66	97.66	203.32	204.92	208.92	205.11
BGDMW	q_1=0.976, q_2=0.973,ν=0.765, θ=2.327,c=1.61	73.89	157.77	160.27	164.78	160.02

and

$$\begin{aligned}\sum_{t\leq y} f(x,t) &= \sum_{t\leq y} S(x,t) - S(x+1,t) - S(x,t+1) + S(x+1,t+1)\\ &= S(x,0) - S(x+1,0) - S(x,y+1) + S(x+1,y+1)\\ &= (1-q_1^{(x+1)^\theta c^{(x+1)}})^\nu - (1-q_1^{x^\theta c^x})^\nu\\ &+ (1-q_1^{x^\theta c^x} q_2^{(y+1)^\theta c^{(y+1)}})^\nu - (1-q_1^{(x+1)^\theta c^{(x+1)}} q_2^{(y+1)^\theta c^{(y+1)}})^\nu.\end{aligned}$$

Thus

$$F_{Y/X=x}(y) = 1 - \frac{(1-q_1^{(x+1)^\theta c^{(x+1)}} q_2^{(y+1)^\theta c^{(y+1)}})^\nu - (1-q_1^{x^\theta c^x} q_2^{(y+1)^\theta c^{(y+1)}})^\nu}{(1-q_1^{(x+1)^\theta c^{(x+1)}})^\nu - (1-q_1^{x^\theta c^x})^\nu}.$$

The random variable (X, Y) is obtained by inverting these equations successively. Here we study the performance of the MLEs of bivariate generalized discrete modified Weibull distribution with parameters $q_1 = 0.67, q_2 = 0.32, \theta = 0.878, c = 1.1$ and $\nu = 0.9$ using Monte Carlo simulation for various sample sizes and for selected parameter values. Now we choose the parameter values as q_1, q_2, θ, c and ν. After simulating the samples we estimate the parameters. Also the bias is obtained for each case. The estimated parameter values and their bias are given in Table 5.

7 Conclusion

We introduced a new bivariate generalized discrete modified Weibull distribution. Here we used a new method to find the joint survival function of the distribution and

Table 5: Simulation study for the parameter set: $q_1 = 0.67$, $q_2 = 0.32$, $\theta = 0.878$, $c = 1.1$ and $\nu = 0.9$.

Sample Size		$q_1 = 0.67$	$q_2 = 0.32$	$\theta = 0.878$	$c = 1.1$	$\nu = 0.9$
n=40	estimate	0.75997	0.508619	1.290315	1.188987	0.826156
	bias	-0.089973	-0.188619	-0.412315	-0.088987	0.073844
	MSE	0.0081	0.035577	0.170003	0.007919	0.005453
n=80	estimate	0.750246	0.477842	1.26366	1.133014	0.848702
	bias	-0.080246	-0.157842	-0.38566	-0.033014	0.051298
	MSE	0.006439	0.024914	0.148734	0.00109	0.002631
n=120	estimate	0.740696	0.47091	1.255584	1.127298	0.8590
	bias	-0.070696	-0.15091	-0.377584	-0.027298	0.041
	MSE	0.004998	0.022774	0.14257	0.000745	0.001681
n=160	estimate	0.740508	0.462928	1.133011	1.117299	0.8692
	bias	-0.07059	-0.142928	-0.255011	-0.017299	0.0308
	MSE	0.004971	0.020428	0.065031	0.000299	0.00095

derived its joint probability mass function of the distribution. Some properties of the distribution are derived in this work. Method of maximum likelihood estimation was used to find the estimates of the parameters. Real life data sets were used to explain the suitability of the distribution. Simulation study is also carried out.

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