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# On Explicit Expressions for Single and Product Moments of Generalized Order Statistics from a New Class of Exponential Distributions and a Characterization

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**ABSTRACT**

In this paper, we establish exact expressions for single and product moments of generalized order statistics from a new family of exponential distributions. We also give a characterization result for this class of distributions.

**Key words and Phrases:** *Generalized order statistics; single and product moments; Frechet-type, Gumble and modified extreme value disributions; characterization*

## 1 Introduction

Generalized order statistics (gos) have been introduced and extensively studied in Kamps (1995 a,b) as a unified theoretical set-up which contains a variety of models of ordered random variables with different interpretations. Examples of such models are: Ordinary order statistics, Sequential order statistics, Progressive type II censored order statistics, Record values, kth record value and Pfeifer's records.

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There is no natural interpretation of generalized order statistics in terms of observed random samples but these models can be effectively applied in life testing and reliability analysis, medical and life time data, and models related to software reliability analysis. The common approach makes it possible to define several distributional properties at once. The structural similarities of these models are based on the similarity of their joint density function.

Let $\{X_n, n \geq 1\}$be a sequence of absolutely continuous, independent and identically distributed random variables with cdf $F(x) = P\ (X \leq x\ )$ and pdf f(x). Assume k $> 0$, n$\in$\{2,3,...\}, $\tilde{m} = (\,m_1\,,\, m_2\,,\, ...,\, m_{n-1})\in R^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{\,1, 2, ..., n-1\,\}$. Then $X\,(\,r,\, n,\, \tilde{m},\, k\,)$, r = 1, 2,...,n , are called gos if their joint pdf is given by

$$f^{X(1,n,\tilde{m},k),\ X(2,n,\tilde{m},k),\ ...,X(n,n,\tilde{m},k)}\,(\,x_1,\, x_2, ..., x_n\,) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)$$
$$\left(\prod_{i=1}^{n-1}(1 - F(x_i))^{m_i}\, f(x_i)\right)(1 - F(x_n))^{k-1}\, f(x_n), \tag{1.1}$$

where $F^{-1}\,(0+)\ < x_1\ \leq x_2 \leq ...\ \leq x_n < F^{-1}(1)$.

By choosing appropriate values of parameters, we get the distribution of a few very common statistics as shown in Table 1.1 given below.

The joint pdf of first r, gos is given by :

$$f^{X(1,n,\tilde{m},k),\ X(2,n,\tilde{m},k),\ ...,\ X(r,n,\tilde{m},k)}\ \ (\,x_1,\, x_2, ..., x_r\,) = c_{r-1}\left(\prod_{i=1}^{r-1}(1 - F(x_i))^{m_i}\, f(x_i)\right)$$
$$(1 - F(x_r))^{k+n-r+M_r-1}\, f(x_r), \tag{1.2}$$

where $F^{-1}\,(0+)\ < x_1\ \leq x_2 \leq ...\ \leq x_r < F^{-1}(1)$.

We now consider two cases:

Case I: $m_1 = m_2 = ... = m_{n-1} = m$

Case II: $\gamma_i \neq \gamma_j$; $i \neq j$, i , j = 1, 2, . . . , n - 1.

For case I, the gos will be denoted by X(r, n, m, k). The pdf of X(r, n, m, k) is given by

$$f^{X(r,n,m,k)}\,(\,x\,) = \frac{c_{r-1}}{(r-1)!}\ \ (1 - F(x))^{\gamma_r - 1}\, f(\,x\,)\ g_m^{r-1}\,(F(x))\,,\ \ x \in R, \tag{1.3}$$

Table 1.1:

| S.No. | Choice of parameters for i = 1, 2,. . . , n | gos becomes |
|---|---|---|
| 1 | $\gamma_i = n - i + 1,\ m_1 = m_2 = ... = m_{n-1} = 0$ and k = 1 | Joint distribution of n order statistics |
| 2 | $\gamma_i = k,\ m_1 = m_2 = ... = m_{n-1} = -1\,,\ k \in N$ | kth record value |
| 3 | $\gamma_i = (n-i+1)\alpha_i\,,\ \alpha_i > 0$ | Sequential order statistics |
| 4 | $\gamma_i = \alpha - i + 1\,,\ \alpha > 0$ | Order statistics with non integer sample size |
| 5 | $\gamma_i = \beta_i\ ,\ \beta_i > 0$ | Pfeifer's record values |
| 6 | $m_i \in N_o\ ,\ k \in N$ | Progressively type-II right censored order statistics |

and the joint pdf of X(r, n, m, k) and X(s, n, m, k), $1 \leq r < s \leq n$, is given by

$$f^{X(r,\,n,m,\,k),\,X(s,\,n,m\,,k)}(x,\ y) = \frac{c_{s-1}}{(r-1)!\ (s-r-1)!}\,\left((1-F(x))^m\, f(x)\right)\ g_m^{r-1}(F(x))$$

$$[h_m\,(F(y)) - h_m\,(F(x))]^{\,s-r-1}\ (1-F(y))^{\gamma_s - 1}\, f(y)\quad ,\ x < y, \tag{1.4}$$

where $c_{r-1} = \prod_{j=1}^{r} \gamma_j\,,\qquad \gamma_j = k + (n-j)(m+1)\ ,\ r = 1,\ 2, ...,\ n\,,$
$g_m(x) = h_m(x) - h_m(0)\,,\ x \in\ (0,\,1)$ and

$$h_m(x) = \begin{cases} -\ \frac{(1-x)^{m+1}}{m+1}\ , & m\ \neq -1, \\ -\ \log\,(1-x)\ , & m\ = -1. \end{cases} \tag{1.5}$$

For case II, the gos will be denoted by $X\,(r,\,n,\,\tilde{m},\,k\,)$. The pdf of $X\,(r,\,n,\,\tilde{m},\,k\,)$ is given by

$$f^{X(r,n,\tilde{m},k)}(x) = c_{r-1}\ f(x)\ \sum_{i=1}^{r} a_i(r)\,(1-F(x))^{\gamma_i - 1}\ ,\quad x \in R, \tag{1.6}$$

and the joint pdf of $X\,(\,r,\ n,\,\tilde{m},\ k\,)$ and $X\,(\,s,\ n,\tilde{m}\,,\,k\,)$, $1 \le r < s \le n$, is given by

$$f^{X(\,r,\,n,\tilde{m},\,k\,),\,X(\,s,\,n,\tilde{m}\,,k\,)}\quad(\,x,\ y\,) = c_{\,s\,-\,1}\ \left\{ \sum_{i=r+1}^{s} a_i^r(s) \left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i} \right\}$$
$$\left\{\sum_{i=1}^{r} a_i(r)\,(1-F(x))^{\gamma_i}\right\} \frac{f(x)}{1-F(x)}\ \frac{f(y)}{1-F(y)}\ , \qquad (1.7)$$

where $c_{\,s\,-\,1} = \prod\limits_{j\,=\,1}^{s} \gamma_j\,, \qquad \gamma_j = k+n-j+m_j\ ,\ s=1,\ 2, ...,\ n.$
Further, it can be proved that

(i) $a_i(r) = \prod\limits_{j\ (\neq i)\ =1}^{r} \ (\gamma_j - \gamma_i)^{-1} \quad , \quad 1 \le i \le r \le n$

(ii) $a_i^r(s) = \prod\limits_{j\ (\neq i)\ =r+1}^{s} \ (\gamma_j - \gamma_i)^{-1} \quad , \quad r+1 \le i \le s \le n$

(iii) $a_i(r) = (\gamma_{r+1} - \gamma_i)\ \ a_i(r+1)$

(iv) $c_r\ = c_{r-1}\gamma_{r+1}$

(v) $\sum_{i\,=\,1}^{r\,+\,1} a_i(r+1) = 0.$

The moments of order statistics have generated considerable interest in the recent years. Several recurrence relations and identities satisfied by single as well as product moments of order statistics have been obtained by various authors in the past. These relations help in reducing the quantum of computations involved. Joshi (1978, 1982) established recurrence relations for exponential distribution with unit mean and were further extended by Balakrishnan and Joshi (1984) for doubly truncated exponential distribution. For linear-exponential distribution, Balakrishnan and Malik (1986) derived the similar type of relations which were extended to doubly truncated linear-exponential distribution by Mohie El-Din et al. (1997) and Saran and Pushkarna (1999). Nain (2010 a, b) obtained recurrence relations for ordinary order statistics and kth record values from pth order exponential and generalized Weibull distributions, respectively.

The recurrence relations for the moments of generalized order statistics based on non identically distributed random variables were developed by Kamps (1995 a, b). Pawlas and Szynal (2001) obtained recurrence relations for single and product moments of generalized order statistics from Pareto, generalized Pareto and Burr distributions. Saran and Pandey (2004, 2009) established recurrence relations for single and product moments of generalized order statistics from linear-exponential and Burr distributions. Saran and Pandey (2011) obtained recurrence relations for marginal and joint moment generating functions of dual (lower) generalized order statistics from inverse Weibull distribution.

In this paper, we derive exact expressions for single and product moments of generalized order statistics for a new class of exponential distributions defined below in Section 2, and discuss its various particular cases. We also give a characterization result for this class of distributions. The results so obtained are generalized versions of Khan et al. (2012).

## 2 A New Family of Exponential Distributions

Consider a family of exponential distributions defined by the function

$$F(x) = 1 - \left(1 - e^{-\Psi(x)}\right)^{\eta}, \eta > 0 \ and \ 0 < x < \infty, \tag{2.1}$$

where$\Psi(0) = \infty$ , $\Psi(\infty) = 0$ and $\Psi(x)$is monotonic in nature with inverse function $\phi(x)$, i.e. , $\Psi^{-1} = \varphi$.

The distribution has many applications in the area of equity risks, extreme floods, the amounts of large insurance losses, the size of freak waves, mutational events during evolution, large wildfires, pipeline failures due to pitting corrosion, etc. The table 2.1 given below demonstrates a few standard distributions obtained from (2.1) by choosing appropriate value of the parameter $\eta$ and the function $\Psi(x)$.

The mathematical form of the distribution function, as given in (2.1), is very useful to derive the exact expressions for single and product moments of gos.

**Notations**

For n = 1, 2, 3,... , a > 0, b > 0, c > 0, $1 \leq r < s \leq n$ , $k \geq 1 \ and \ u, v \in \{\ 0, 1, 2, ...\}$,

Table 2.1:

| S.No. | Choice of parameter $\eta$ and the function $\Psi(x)$ | Family of exponential distribution represents |
|---|---|---|
| 1 | $\eta = 1,\ \Psi(x) = \left(\frac{\beta}{x}\right)^{\alpha},\ \phi(x) = \beta\, x^{-\frac{1}{\alpha}},\ 0 < x < \infty\ and\ \beta > 0.$ | Frechet-type extreme value distribution |
| 2 | $\eta = 1,\ \Psi(x) = e^{-\lambda x},\ \phi(x) = -\frac{1}{\lambda}\log x,\ 0 < x < \infty\ and\ \lambda > 0.$ | Gumbel Extreme value distribution |
| 3 | $\eta = 1,\ \Psi(x) = e^{-(\lambda x)^{\frac{1}{\alpha}}},\ \phi(x) = \frac{1}{\lambda}(-\log x)^{\alpha},\ 0 < x < \infty\ and\ \alpha, \lambda > 0.$ | Modified extreme value Type-I distribution |

we denote by

1. $$H_u\,(a,b) = \int_0^{\infty} x^u\,(1 - F(x))^a\, f\,(x)\, g_m^b\,(F(x))\,dx \tag{2.2}$$
2. $$\begin{aligned} H_{u\,,\,v}\,(a,b,c) &= \int_0^{\infty}\int_x^{\infty} x^u\, y^v\,(1 - F(x))^a\, f\,(x) \\ &\times [h_m\,(F(y)) - h_m\,(F(x))]^{\,b}\,(1 - F(y))^c\, f(y)\,dy\,dx \end{aligned} \tag{2.3}$$
3. $$\mu^u_{m,\ n,\ k}\,(r) = E\,(\mathrm{X}\,(\ \mathrm{r},\ \mathrm{n},\ \mathrm{m},\ \mathrm{k}\ ))^u \tag{2.4}$$
4. $$\mu^{u,\ v}_{m,\ n,\ k}\,(r,s) = E\,((\mathrm{X}\,(\ \mathrm{r},\ \mathrm{n},\ \mathrm{m},\ \mathrm{k}\ )\,)^u\,(\,\mathrm{X}\,(\ \mathrm{s},\ \mathrm{n},\ \mathrm{m},\ \mathrm{k}\ ))^v) \tag{2.5}$$
5. $$\mu^u_{\tilde{m},\ n,\ k}\,(r) = E\,[\mathrm{X}\,(\ \mathrm{r},\ \mathrm{n},\ \tilde{\mathrm{m}},\ \mathrm{k}\ )]^u \tag{2.6}$$
6. $$\mu^{u,\ v}_{\tilde{m},\ n,\ k}\,(r,s) = E\,[(\mathrm{X}\,(\ \mathrm{r},\ \mathrm{n},\ \tilde{\mathrm{m}},\ \mathrm{k}\ )\,)^u\,(\mathrm{X}\,(\ \mathrm{s},\ \mathrm{n},\ \tilde{\mathrm{m}},\ \mathrm{k}\ ))^v] \tag{2.7}$$

## 3 Some Auxiliary Results

In this section, we establish some results which will be useful later for deriving the main results.

**Lemma 3.1** For the class of distributions defined in (2.1) and non-negative finite integers $i$, $j$, $a$ and $b$,

$$H_i(a,b) = \begin{cases} \frac{1}{(m+1)^b} \sum_{d=0}^{b} \sum_{w=0}^{\infty} \binom{b}{d} (-1)^d \frac{\beta_w(i)}{\frac{w+i}{\eta}+a+(m+1)d+1}, & m \neq -1, \\ b! \sum_{w=0}^{\infty} \frac{\beta_w(i)}{\left(\frac{w+i}{\eta}+a+1\right)^{b+1}}, & m = -1, \end{cases}$$

where $\beta_w(i)$is the co-efficient of $t^{w+i}$ in $\left[\sum_{w=0}^{\infty} \frac{\phi_w(0)}{w!} \left(\sum_{s=1}^{\infty} \frac{t^s}{s}\right)^w\right]^i$ and $\phi = \Psi^{-1}$ as defined earlier.

**Proof** .

Case 1. $m \neq -1$.

Substituting $g_m^b(F(x)) = \left(\frac{1-(1-F(x))^{m+1}}{m+1}\right)^b = \frac{1}{(m+1)^b} \sum_{d=0}^{b} \binom{b}{d} (-1)^d (1-F(x))^{(m+1)d}$ in (2.2), we get

$$H_i(a,b) = \frac{1}{(m+1)^b} \sum_{d=0}^{b} \binom{b}{d} (-1)^d \int_0^{\infty} x^i (1-F(x))^{a+(m+1)d} f(x)\, dx.$$

Putting $t = (1-F(x))^{\frac{1}{\eta}}$, we have

$$H_i(a,b) = \frac{\eta}{(m+1)^b} \sum_{d=0}^{b} \sum_{w=0}^{\infty} \binom{b}{d} (-1)^d \beta_w(i) \int_0^1 t^{w+i+(a+(m+1)d+1)\,\eta-1}\, dt$$

$$= \frac{\eta}{(m+1)^b} \sum_{d=0}^{b} \sum_{w=0}^{\infty} \binom{b}{d} (-1)^d \frac{\beta_w(i)}{w+i+(a+(m+1)d+1\,)\,\eta},$$

which leads to the relation as stated in Lemma 3.1 for the case $m \neq -1$.

Case 2. $m = -1$

By using repeatedly the combinatorial identity (see, Ruiz, 1996)

$$\sum_{d=0}^{b} \binom{b}{d} (-1)^d\ d^k = \begin{cases} 0, & k = 0,1,2,...,b-1, \\ (-1)^b\, b!, & k = b, \end{cases} \tag{3.1}$$

we get

$$H_i(a,b) = \lim_{m \to -1} \sum_{w=0}^{\infty} \beta_w(i) \frac{\sum_{d=0}^{b} \binom{b}{d} (-1)^d \times \left(\frac{w+i}{\eta} + a + (m+1)d + 1\right)^{-1}}{(m+1)^b}$$

$$= \sum_{w=0}^{\infty} \beta_w(i) \left( \frac{w+i}{\eta} + a + 1 \right)^{-b-1} \times \sum_{d=0}^{b} \begin{pmatrix} b \\ d \end{pmatrix} (-1)^{d+b} \times d^b$$

(by using L-Hospital Rule),

which again on using (3.1) for k = b leads to the relation as stated in Lemma 3.1 for the case $m = -1$.

**Lemma 3.2** For the class of distributions (2.1) and non-negative finite integers $i, j, a, b$ and $c$,

$$H_{i,j}(a,b,c) = \begin{cases} \frac{1}{(m+1)^b} \sum_{v=0}^{b} \sum_{w=0}^{\infty} \sum_{w'=0}^{\infty} \begin{pmatrix} b \\ v \end{pmatrix} (-1)^v \times \frac{\beta_w(i)}{\frac{w+w'+i+j}{\eta} + (a+c+b(m+1)+2)} \\ \qquad \times \frac{\beta_{w'}(j)}{\frac{w'+j}{\eta} + ((b-v)(m+1)+c+1)}, & m \neq -1, \\ b! \sum_{w=0}^{\infty} \sum_{w'=0}^{\infty} \frac{\beta_w(i)}{\left(\frac{w+w'+i+j}{\eta} + a + c + 2\right)^{b+1}} \times \frac{\beta_{w'}(j)}{\left(\frac{w'+j}{\eta} + a + 1\right)^{b+1}}, & m = -1, \end{cases}$$

where $\beta_w(i)$is as defined in Lemma 3.1.

**Proof**

Case 1: $m \neq -1$.

From (2.3), we have, for $b = 0$,

$$H_{i,j}(a,0,c) = \int_0^{\infty} \int_x^{\infty} x^i y^j (1-F(x))^a f(x) (1-F(y))^c f(y)\, dy\, dx$$

$$= \int_0^{\infty} x^i (1-F(x))^a f(x)\, G(x)\, dx, \tag{3.2}$$

where

$$G(x) = \int_x^{\infty} y^j (1-F(y))^c f(y)\, dy. \tag{3.3}$$

Putting $t = (1-F(y))^{\frac{1}{\eta}}$ in (3.3), we have

$$G(x) = \eta \int_0^{(1-F(x))^{\frac{1}{\eta}}} \left( \sum_{w'=0}^{\infty} \beta_{w'}(j)\ t^{j+w'} \right) t^{\eta c+\eta-1} dt$$

$$= \eta \sum_{w'=0}^{\infty} \beta_{w'}(j) \int_0^{(1-F(x))^{\frac{1}{\eta}}} t^{w'+j+\eta(1+c)-1} dt$$

$$= \sum_{w'=0}^{\infty} \beta_{w'}(j) \left[ \frac{(1-F(x))^{\frac{w'+j}{\eta}+c+1}}{\frac{w'+j}{\eta}+c+1} \right],$$

which on substituting in (3.2) gives,

$$H_{i\,,\,j}(a,0,c) = \eta^2 \sum_{w=0}^{\infty} \sum_{w'=0}^{\infty} \beta_w(i)\, \beta_{w'}(j) \; \frac{1}{w'+j+(c+1)\eta} \int_0^1 t^{w+w'+i+j+\eta(a+\,c+2)-1}\, dt$$

$$= \eta^2 \sum_{w=0}^{\infty} \sum_{w'=0}^{\infty} \frac{\beta_w(i)}{w+w'+i+j+(a+\,c+2)\,\eta} \times \frac{\beta_{w'}(j)}{w'+j+(c+1)\,\eta}\,. \qquad (3.4)$$

Further, on substituting the value of

$$[h_m(F(y)) - h_m(F(x))]^b = \left[ \frac{(1-F(x))^{m+1} - (1-F(y))^{m+1}}{(m+1)} \right]^b$$

$$= \frac{1}{(m+1)^b} \sum_{v=0}^{b} \binom{b}{v} (-1)^v \; (1-F(x))^{v(m+1)} (1-F(y))^{(b-v)(m+1)}$$

in (2.3), we get

$$H_{i\,,\,j}(a,b,c) = \frac{1}{(m+1)^b} \sum_{v=0}^{b} \binom{b}{v} (-1)^v$$

$$\times \left[ \int_0^{\infty} \int_x^{\infty} x^i\, y^j\, (1-F(x))^{a+v(m+1)} f(x)\, (1-F(y))^{(b-v)(m+1)+c} f(y)\, dy\, dx \right]$$

$$= \frac{1}{(m+1)^b} \sum_{v=0}^{b} \binom{b}{v} (-1)^v H_{i\,,\,j}(a+v(m+1),\; 0\;, (b-v)(m+1)+c)\,,$$

(by using (3.2))

which on using (3.4) leads to the relation as stated in Lemma 3.2 for the case $m \neq -1$.

Case 2. $m = -1$.

The proof is similar to the one used in case 2 of Lemma 3.1.

# 4 Explicit Expressions For Single and Product Moments

**Theorem 4.1** For $m_1 = m_2 = ... = m_{n-1} = m$, n = 1, 2, 3,... , $1 \leq r \leq n\,,\ k \geq 1$ and $u \in \{\ 0, 1, 2, ...\}$,

$$\mu^{u}_{m,\ n,\ k}\,(r) = \frac{c_{\,r\,-\,1}}{(r-1)!}\,H_u\,(\gamma_r - 1,\ r-1)\,, \tag{4.1}$$

where $H_u\,(\gamma_r - 1,\ r-1)$ is as defined in Lemma 3.1.

**Proof.** On using (2.4) and (1.3), the $uth$ order moment of $X(r, n, m, k)$ is given by

$$\mu^{u}_{m,\ n,\ k}\,(r) = \frac{c_{\,r\,-\,1}}{(r-1)!}\ \int_0^\infty x^u\,(1-F(x))^{\gamma_r - 1}\,f(\,x\,)\ g_m^{r\,-\,1}\,(F(x))\,dx.$$

By using (2.2), we shall derive the relation as stated in (4.1).

**Corollary 4.1** For $\gamma_i \neq \gamma_j\ ; i \neq j,\ \ i, j = 1, 2, ..., n-1,\ n = 1, 2, 3, ...,\ 1 \leq r \leq n\,,\ k \geq 1$ and

$$u \in \{\ 0, 1, 2, ...\}\,,$$

$$\mu^{u}_{\tilde{m},\ n,\ k}\,(r) = c_{\,r\,-\,1}\sum_{i\,=\,1}^{r} a_i(r)\,H_u\,(\gamma_i - 1, 0)\,, \tag{4.2}$$

where $H_u\,(\gamma_i - 1, 0)$ is as defined in Lemma 3.1.

**Proof.** The proof is similar to the proof of theorem 4.1 on using (2.6) and (1.6).

**Theorem 4.2** For $m_1 = m_2 = ... = m_{n-1} = m$, n = 1, 2, 3,... , $1 \leq r < s \leq n\,,\ k \geq 1$ and $u, v \in \{\ 0, 1, 2, ...\}$,

$$\mu^{u,\ v}_{m,\ n,\ k}\,(r, s) = \frac{c_{\,s\ -1}}{(r-1)!\ (s-r-1)!\ \ (m+1)^{r-1}}\sum_{d=0}^{r-1}\left(\begin{array}{c} r-1 \\ d \end{array}\right)(-1)^d$$

$$\times\, H_{u\,,\,v}\,(m + (m+1)d,\ \ s-r-1,\ \ \gamma_s - 1)\,, \tag{4.3}$$

where $H_{u\,,\,v}\,(m + (m+1)d,\ \ s-r-1,\ \ \gamma_s - 1)$ is as defined in Lemma 3.2.

**Proof.** On using (2.5) and (1.4), we have

$$\mu^{u,\ v}_{m,\ n,\ k}\,(r, s) = \frac{c_{\,s\ -1}}{(r-1)!\ (s-r-1)!}\int_0^\infty \int_x^\infty x^u\,y^v\ \left(\,(1-F(x))^m\,f(x)\right)\ g_m^{r\,-\,1}\,(F(x))$$

$$\times\ \left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}\left(1-F(y)\right)^{\gamma_s-1}f(y)\,dy\,dx. \qquad (4.4)$$

Substituting

$$\begin{aligned} g_m^{r-1}\left(F(x)\right) &= \left(\frac{1-\left(1-F(x)\right)^{m+1}}{m+1}\right)^{r-1} \\ &= \frac{1}{(m+1)^{r-1}}\sum_{d=0}^{r-1}\begin{pmatrix} r-1 \\ d \end{pmatrix}(-1)^d\left(1-F(x)\right)^{(m+1)d} \end{aligned}$$

in (4.4), we have

$$\begin{aligned} \mu_{m,\ n,\ k}^{u,\ v}(r,s) &= \frac{c_{s\ -1}}{(r-1)!\ (s-r-1)!\ (m+1)^{r-1}} \\ & \sum_{d=0}^{r-1}\begin{pmatrix} r-1 \\ d \end{pmatrix}(-1)^d\int_0^\infty\int_x^\infty x^u\,y^v\left(\left(1-F(x)\right)^{m+(m+1)d}f(x)\right) \\ & \times\ \left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}\left(1-F(y)\right)^{\gamma_s-1}f(y)\,dy\,dx. \end{aligned}$$

The relation (4.3) follows immediately on using (2.3).

**Corollary 4.2** For $\gamma_i\neq\gamma_j$ ; $i\neq j$, $i,j=1,2,...,n-1$**,** n = 1, 2, 3,... , $1\leq r<s\leq n$ ,, $k\geq 1$ and $u,v\in\{\ 0,1,2,...\}$,

$$\mu_{\tilde{m},n,k}^{u,v}(r,s)=c_{s-1}\sum_{i\ =\ 1}^{r}a_i(r)\sum_{j\ =\ r+1}^{s}a_j^r(s)\ H_{i\,,\,j}\left(\gamma_i-\gamma_j-1,0,\gamma_j-1\right), \qquad (4.5)$$

where $H_{i\,,\,j}\left(\gamma_i-\gamma_j-1,0,\gamma_j-1\right)$is as defined in Lemma 3.2.

**Proof.** The proof is similar to the proof of theorem 4.2 on employing (2.7) and (1.7).

## 5 Characterization

Let $X(r,n,m,k)$, $r=1,2,\ldots,n$ be the gos from a continuous type of distribution with cumulative distribution function $F(x)$ and probability density function $f(x)$. Then, in view of (1.3) and (1.4), the conditional density function of $Y=X(s,n,m,k)$ given $X(r,n,m,k)=x$, $1\leq r<s\leq n$, is

$$f(y|x) = \sigma \left[1 - \left(\frac{1-F(y)}{1-F(x)}\right)^{m+1}\right]^{s-r-1} \left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_s - 1} \frac{f(y)}{1-F(x)} \; , \;\; 0 < x < y < \infty, \tag{5.1}$$

where $\sigma = \frac{c_{s-1}}{(s-r-1)!\; c_{r-1}\; (m+1)^{s-r-1}}$.

**Theorem 5.1** Let X be a non-negative, absolutely continuous type of random variable with distribution function F(x) satisfying the conditions $F(0) = 0$ and $0 < F(x) < 1$. Then a necessary and sufficient condition for

$$E\left((X\,(\,s,\; n, m\,,\, k\,))^i\,|X\,(\,r,\; n, m\,,\, k\,) = x\right) \;=\; \sum_{w=0}^{\infty} \beta_w\,(i)\,\left(1 - e^{-\psi(x)}\right)^{w+i} \times \prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\frac{w+i}{\eta} + \gamma_{r+j}}\right) \tag{5.2}$$

is that

$$F\,(x) = 1 - \left(1 - e^{-\psi(x)}\right)^{\eta}, \; x > 0\,,\; \eta > 0,$$

where $\psi(x)$ is monotonic function satisfying $\psi o \varphi(x) = x$ for some function $\varphi(x)$.

**Proof.** On using (5.1), we have

$$E\left(Y^i|X\,(\,r,\; n, m\,,\, k\,) = x\right)$$
$$= \sigma \int_x^{\infty} y^i \left(1 - \left(\frac{1-F(y)}{1-F(x)}\right)^{m+1}\right)^{s-r-1} \left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_s - 1} \frac{f(y)}{1-F(x)}\, dy$$

Put $u = \frac{1-F(y)}{1-F(x)}$ in the above equation we get

$$\begin{aligned}
&E\left(Y^i|X\,(\,r,\; n, m\,,\, k\,) = x\right) \\
&= \sigma \sum_{w=0}^{\infty} \beta_w\,(i)\,\left(1 - e^{-\psi(x)}\right)^{w+i} \int_0^1 u^{\frac{w+i}{\eta} + \gamma_s - 1}\,\left(1 - u^{m+1}\right)^{s-r-1} du, \\
&= \sigma \sum_{w=0}^{\infty} \beta_w\,(i)\,\left(1 - e^{-\psi(x)}\right)^{w+i}\; B\left(\frac{w+i}{\eta(m+1)} + \frac{\gamma_s}{m+1}, s-r\right) (By\ putting\ u^{m+1} = v) \\
&= \frac{c_{s-1}}{c_{r-1}} \sum_{w=0}^{\infty} \beta_w\,(i)\,\left(1 - e^{-\psi(x)}\right)^{w+i} \prod_{j=1}^{s-r}\left(\frac{w+i}{\eta} + \gamma_{r+j}\right)^{-1},
\end{aligned}$$

which, on substituting the value of $\frac{c_{s-1}}{c_{r-1}}$, leads to (5.2). This proves the sufficient condition.

Let

$$Z_r(x) = \sum_{w=0}^{\infty} \beta_w\,(i)\,\left(1 - e^{-\psi(x)}\right)^{w+i} \prod_{j=1}^{s-r} \left( \frac{\gamma_{r+j}}{\frac{w+i}{\eta} + \gamma_{r+j}} \right). \tag{5.3}$$

Then it implies that

$$Z_{r+1}(x) - Z_r(x) = \frac{1}{\eta\,\gamma_{r+1}} \sum_{w=0}^{\infty} \beta_w\,(i)\,(w+i)\,\left(1 - e^{-\psi(x)}\right)^{w+i} \prod_{j=1}^{s-r} \left( \frac{\gamma_{r+j}}{\frac{w+i}{\eta} + \gamma_{r+j}} \right). \tag{5.4}$$

Differentiating both sides of (5.3) with respect to $x$, we have

$$Z_r^{'}(x) = \frac{e^{-\psi(x)}\psi'(x)}{1 - e^{-\psi(x)}} \sum_{w=0}^{\infty} \beta_w\,(i)\,(w+i)\left(1 - e^{-\psi(x)}\right)^{w+i} \prod_{j=1}^{s-r} \left( \frac{\gamma_{r+j}}{\frac{w+i}{\eta} + \gamma_{r+j}} \right). \tag{5.5}$$

Using (5.4) and (5.5), we get

$$Z_r^{'}(x) = \gamma_{r+1}\,(Z_{r+1}(x) - Z_r(x))\,\frac{\eta\;\; e^{-\psi(x)}\psi'(x)}{1 - e^{-\psi(x)}}. \tag{5.6}$$

Also from (5.2), we have

$$\frac{c_{s-1}}{c_{r-1}} \int_x^{\infty} y^i\;\left((1-F(x))^{m+1} - (1-F(y))^{m+1}\right)^{s-r-1} (1-F(y))^{\gamma_s - 1}\,f(y)\,dy$$

$$= (s-r-1)!\,(m+1)^{s-r-1}\;\,(1-F(x))^{\gamma_{r+1}}\,Z_r(x). \tag{5.7}$$

Differentiating both sides with respect to x, we get

$$\frac{c_{s-1}}{c_{r-1}} \left[ \int_x^{\infty} y^i\;\left((1-F(x))^{m+1} - (1-F(y))^{m+1}\right)^{s-r-2} (1-F(y))^{\gamma_s - 1}\,f(y)\,dy \right]$$

$$= \frac{(s-r-2)!\;\;(m+1)^{s-r-2}\;\,(1-F(x))^{\gamma_{r+1}-1}}{f(x)\,(1-F(x))^m} \left( \gamma_{r+1}\,f(x) Z_r(x) + (1-F(x)) Z_r^{/}(x) \right). \tag{5.8}$$

Using (5.7) in the L.H.S. of (5.8) by making appropriate changes we get

$$\frac{c_{s-1}}{c_{r-1}} \left( \frac{c_r}{c_{s-1}} (s-r-2)!\;\;(m+1)^{s-r-2}\,(1-F(x))^{\gamma_{r+2}}\,Z_{r+1}(x) \right)$$

$$= (s-r-2)!\;\;(m+1)^{s-r-2}\,(1-F(x))^{\gamma_{r+2}} \left( \gamma_{r+1}\,Z_r(x) + \frac{1-F(x)}{f(x)} Z_r^{/}(x) \right),$$

which on simplification yields

$$\gamma_{r+1}\left(Z_{r+1}(x) - Z_r(x)\right) = \frac{1-F(x)}{f(x)} Z_r^{'}(x). \tag{5.9}$$

Then, using (5.6) in (5.9) we get

$$\frac{f(x)}{1-F(x)} = \frac{\eta \;\; e^{-\psi(x)}\psi'(x)}{1-e^{-\psi(x)}}$$

This implies

$$F\,(x) = 1 - \left(1 - e^{-\psi(x)}\right)^{\eta}, \quad x > 0\,, \eta > 0.$$

This proves the necessary part and hence the result.

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