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Estimation of the Scale Parameter of Triangular Distribution Using Absolved Order Statistics

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ABSTRACT

In this paper, a new set of ordered random variables called Absolved Order Statistics(AOS) from a scale dependent triangular distribution is considered. The vector of AOS is found to be the minimal sufficient statistic for the triangular distribution with scale parameter σ. Based on AOS, the best linear unbiased estimate $\hat{\sigma}$ of σ along with its variance is explicitly derived. It is found that censoring based on AOS is more realistic and the estimate obtained from it for σ is more efficient than the case of censoring with order statistics.In this study, we also obtained the U-statistic estimator based on AOS for the scale parameter σ of the triangular distribution.

Key words and Phrases: *Triangular Distribution; Order Statistics; Absolved Order Statistics; Minimal Sufficient Statistics; Best Linear Unbiased Estimate; Estimation from Censored Samples; U-statistics.*

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1 INTRODUCTION

The scale dependent triangular distribution over $[-\sigma, \sigma]$ is defined by the probability density function (pdf) given by

$$f(x, \sigma) = \frac{1}{\sigma}\Big(1 - \frac{|x|}{\sigma}\Big), -\sigma < x < \sigma, \sigma > 0, \tag{1.1}$$

where σ is the scale parameter of the distribution. Various extensions of the triangular distribution were discussed by Rene van Dorp and Kotz (2002), Glickman and Xu (2008), and Garg et al. (2009). Johnson and Kotz (1999) described situations wherein triangular distribution is used as an alternative model to the beta distribution. Balakrishnan and Nevzorov (2004) studied some properties of the triangular distribution. Triangular distribution find applications in the Project Evaluation and Review Technique as well as Risk analysis. For details see, Johnson (1997), Fairchild et al. (2016) and Johnson (2002).

Let X be a random variable with pdf (1.1), then $E(X) = 0$ and $V(X) = \frac{\sigma^2}{6}$. Kotz and René Van Dorp (2004) discussed about moments of the triangular distribution in detail and also explained the maximum likelihood estimation procedure for the parameter σ of triangular distribution. The best linear unbiased estimation in triangular distribution was carried out using order statistics in Samuel and Thomas (2003). The maximum likelihood (ML) method of estimation of σ in (1.1) is somewhat not so trivial as the likelihood function is not differentiable at σ. The form of the density (1.1) and the likelihood resulting from it fails to provide the ML estimate of σ explicitly. Several authors though tried to obtain the ML estimate, they end up with procedures of obtaining the estimate involved with a software that do not contain detailed descriptions of their procedure. In this respect the estimate provided by Samuel and Thomas (2003) attracts some importance at least in small sample cases.

Thomas and Anjana (2021) have considered the family $\mathscr{F}_1$ of all absolutely continuous distributions which are symmetrically distributed about zero and defined a new variety of ordered random variables and is given below

Definition 1.1. *(Thomas and Anjana, 2021). Suppose* $X_1, X_2, ..., X_n$ *is a random sample of size n drawn from a distribution with pdf* $f(x)$ *such that* $f(x) \in \mathscr{F}_1$*. If we take the absolute values of the observations and order them in the increasing order of magnitude as* $X_{(1:n)} \leq X_{(2:n)} \leq ... \leq X_{(n:n)}$*, then we say that* $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ *are the absolved order statistics (AOS) of the given sample.*

Thomas and Anjana (2021) further proved the following theorem on the minimal sufficiency of the statistics introduced in the above definition.

Theorem 1.1. *(see, Theorem 2.1 of Thomas and Anjana, 2021). Suppose* $\underset{\sim}{X} = (X_1, X_2, ..., X_n)$ *is a vector of observations of a random sample of size n drawn from a distribution with pdf* $f(x) \in \mathscr{F}_1$*. Let* $T = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$ *be a statistic based on the AOS* $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ *constructed from the observations* $X_1, X_2, ..., X_n$*. Then T is minimal sufficient for the family* $\mathscr{F}_1$*.*

The vector of order statistics $U = (X_{1:n}, X_{2:n}, ..., X_{n:n})'$ of a random sample $X_1, X_2, ..., X_n$ of size n arising from a distribution belonging to the family $\mathscr{F}$ of absolutely continuous distributions was proved as complete and minimal sufficient statistic for $\mathscr{F}$. As $\mathscr{F}_1 \subset \mathscr{F}$, no thought is seen went through the minds of authors about the improvement of minimal sufficient statistic for the sub-class $\mathscr{F}_1$ of $\mathscr{F}$. The Theorem 1.1 proved by Thomas and Anjana (2021) provided the improvement of the minimal sufficient statistic for $\mathscr{F}_1$. Clearly the pdf (1.1) belongs to $\mathscr{F}_1$ and hence the BLUE based on AOS should have an advantage over the BLUE based on order statistics in the estimation of the parameter σ involved in (1.1).

In Section 2, we derive the distribution theory of AOS arising from (1.1) and proved that vector of AOS is minimal sufficient for the family of triangular distributions as well. In Section 3, we have discussed about the estimate of σ in (1.1) by the BLUE based on AOS. The efficiency of this estimate relative to the usual BLUE based on order statistics for sample sizes $n = 2(1)20$ is computed and analyzed. Since this estimate of σ for triangular distribution is not perceived well, another type of estimator called U-statistic defined from BLUE based on AOS as kernel is attempted in section 5. It is to be noted that U-statistics are unbiased consistent estimators and they are normally distributed as $n \longrightarrow \infty$.

2 MINIMAL SUFFICIENT STATISTIC FOR THE TRIANGULAR DISTRIBUTION

Let $X_1, X_2, ..., X_n$ be a random sample of size n drawn from the triangular distribution with pdf as given in (1.1). Suppose $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ are the AOS of the sample. Then the distribution theory of AOS is given in the following theorem.

Theorem 2.1. *Suppose $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ are the AOS of a random sample of size n drawn from the distribution with pdf $f(x, \sigma)$ which is as given in (1.1). Let $Z_{1:n}, Z_{2:n}, ..., Z_{n:n}$ be the order statistics of a random sample of size n arising from the folded distribution of (1.1) with density $g(z, \sigma) = 2f(z, \sigma), 0 \leq z < \sigma$. Then*

$$(X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}) \stackrel{d}{=} (Z_{1:n}, Z_{2:n}, ..., Z_{n:n}),$$

where $X \stackrel{d}{=} Z$ is the usual notation representing the identically distributed property between two random variables X and Z.

Proof. Let X follow triangular distribution with pdf (1.1). Let $Z = |X|$, then Z with pdf $g(z, \sigma) = 2f(z, \sigma), 0 \leq z < \sigma$, follows half-triangular distribution. Since $X_{(1:n)} \leq X_{(2:n)} \leq ... \leq X_{(n:n)}$ are the order statistics of the random sample $|X_1|, |X_2|, ..., |X_n|$. We have $(X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$ is distributed identically as the vector of order statistics of a random sample of size n arising from half-triangular distribution. Hence the theorem. □

Now, let us discuss about a minimal sufficient statistic for the triangular family of distributions as defined by the pdf (1.1). From theorem 1.1 due to Thomas and Anjana (2021), it is observed that the statistic $T = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$ formed from the AOS of a random sample of size n is the minimal sufficient statistic for the family$\mathscr{F}_1$ of distributions which are all symmetric about zero. If we consider a specific member of $\mathscr{F}_1$ with pdf $f(x)$, then some times it is possible to get a dimensional reduction in the minimal sufficient statistic for $f(x)$. For example, some distributions belonging to the exponential family (such as normal distribution) which are distributed symmetrically about zero, we get a minimal sufficient statistic which is different from $T = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$ and whose dimension is less than n. The essential point is that by merely observing triangular distribution as defined in (1.1) belongs to $\mathscr{F}_1$, we cannot claim that $T = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$ is also minimal sufficient for triangular distribution. Clearly, triangular distribution is not a member of exponential family and similarly we can verify that triangular distribution is not a member to any sub-family of the class of all continuous distributions for which an improved version of minimal sufficient statistic exists in terms of reduced dimension than $T = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$. Observing the above arguments about the irreducible nature of the sufficient statistic than T, we can prove the following theorem using basic principles.

Theorem 2.2. *Suppose* $\underline{X} = (X_1, X_2, ..., X_n)$ *is a random sample of size n drawn from the triangular distribution with pdf* $f(x, \sigma)$ *as given in (1.1). Let* $T = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})$ *be the vector of AOS constructed from* $X_1, X_2, ..., X_n$. *Then* T *is minimal sufficient for* $f(x, \sigma)$.

The proof of this theorem is not given here similar to the proof given by Thomas and Anjana (2021). Hence omitted.

It is well known in statistical inference that any inference procedure developed based on minimal sufficient statistic excel in performance than those based on other statistics. Thus at this stage it is destined to observe that the Best Linear Unbiased Estimate (BLUE) of σ based on AOS is better than that based on the classical order statistics. In the next section this aspect is discussed more.

3 BEST LINEAR UNBIASED ESTIMATION OF THE SCALE PARAMETER BASED ON AOS

There is extensive literature available about the problem of estimation of the location and scale parameters of a distribution by order statistics. For details see, David and Nagaraja (2003), Balakrishnan and Cohen (1991). For modification of the BLUE for symmetric populations see, Thomas (1990). In this section by a similar manner we apply AOS to estimate the scale parameter σ of triangular distribution. The standard form of triangular distribution as given in (1.1) has the pdf

$$f(x) = \Big(1 - |x|\Big), -1 < x < 1. \tag{3.1}$$

The expressions for the single and product moments of order statistics arising from the two parameter triangular distribution along with best linear unbiased estimators of the location and scale parameters based on order statistics were obtained in Samuel and Thomas (2003). From (1.1) the pdf of the half-triangular distribution can be written as

$$g(x,\sigma) = \frac{2}{\sigma}\Big(1 - \frac{x}{\sigma}\Big), 0 \leq x \leq \sigma, \sigma > 0. \tag{3.2}$$

The pdf of the corresponding standard form of half-triangular distribution is

$$g(x,1) = 2\Big(1 - x\Big), 0 \leq x \leq 1. \tag{3.3}$$

In the following theorem, the expression for the BLUE and its variance for the scale parameter σ of the triangular distribution based on AOS is given.

Theorem 3.1. *Let* $\underset{\sim}{X} = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})'$ *be the first n AOS of a random sample of size n drawn from the triangular distribution defined in (1.1). Suppose* $\underset{\sim}{Y} = (Y_{1:n}, Y_{2:n}, ..., Y_{n:n})'$ *is the vector of order statistics arising from the standard half-triangular distribution defined by the pdf (3.3). Let* $E(\underset{\sim}{Y}) = \underset{\sim}{\alpha} = (\alpha_{1:n}, \alpha_{2:n}, ..., \alpha_{n:n})'$ *and let the dispersion matrix of* $\underset{\sim}{Y}$ *be given by* $D(\underset{\sim}{Y}) = A = ((\alpha_{i,j:n}))$*, where* $\alpha_{i,j:n} = Cov(Y_{i:n}, Y_{j:n}), i, j = 1, 2, ..., n,$ *for* $i \neq j$ *and* $\alpha_{i,i:n} = Var(Y_{i:n})$ *for* $i = 1, 2, ..., n$*. Then the BLUE* $\hat{\sigma}$ *of* σ *based on AOS is given by*

$$\hat{\sigma} = (\underset{\sim}{\alpha}'\mathbf{A}^{-1}\underset{\sim}{\alpha})^{-1}\underset{\sim}{\alpha}'\mathbf{A}^{-1}\underset{\sim}{X} \tag{3.4}$$

and its variance is given by

$$Var(\hat{\sigma}) = (\underset{\sim}{\alpha}'\mathbf{A}^{-1}\underset{\sim}{\alpha})^{-1}\sigma^2. \tag{3.5}$$

Proof. Let $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ be the AOS of a random sample of size n drawn from the triangular distribution with pdf (1.1). Then $(\frac{X_{(1:n)}}{\sigma}, \frac{X_{(2:n)}}{\sigma}, ..., \frac{X_{(n:n)}}{\sigma})' \stackrel{d}{=} (Y_{1:n}, Y_{2:n}, ..., Y_{n:n})'$ where $Y_{1:n}, Y_{2:n}, ..., Y_{n:n}$ are the order statistics of a random sample of size n drawn from the standard half-triangular distribution with pdf (3.3).

Clearly the means, variances and covariances of all order statistics involved in the right side vector of above distributional identity exists finitely and they are independent of σ. Thus for $\underset{\sim}{X} = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)})'$ and $\underset{\sim}{Y} = (Y_{1:n}, Y_{2:n}, ..., Y_{n:n})'$ we can write

$$E(\underset{\sim}{X}) = \sigma E(\underset{\sim}{Y}) = \sigma \underset{\sim}{\alpha} \tag{3.6}$$

and

$$D(\underset{\sim}{X}) = \sigma^2 D(\underset{\sim}{Y}) = \sigma^2 A. \tag{3.7}$$

Then (3.6) and (3.7) together form a generalized Gauss-Markov setup and hence by Gauss-Markov theorem the BLUE of σ is given by

$$\hat{\sigma} = (\underset{\sim}{\alpha}' \mathbf{A}^{-1} \underset{\sim}{\alpha})^{-1} \underset{\sim}{\alpha}' \mathbf{A}^{-1} \underset{\sim}{X} \tag{3.8}$$

and

$$Var(\hat{\sigma}) = (\underset{\sim}{\alpha}' \mathbf{A}^{-1} \underset{\sim}{\alpha})^{-1} \sigma^2. \tag{3.9}$$

This proves the theorem. □

The linear estimate $\hat{\sigma}$ of σ as given in (3.8) may also be written as

$$\hat{\sigma} = \sum_{i=1}^{n} c_{i,n} X_{(i:n)}, \tag{3.10}$$

where $c_{i,n}$, $i = 1, 2, ..., n$ are appropriate constants.

In (3.10), it is strange to note that $X_{(i:n)}$, $i = 1, 2, ..., n$ are the AOS arising from the triangular distribution defined in (1.1), while $c_{i,n}$, $i = 1, 2, ..., n$ are the coefficients of BLUE of σ based on the order statistics of a random sample of size n arising from the half-triangular distribution defined by the pdf (3.2)(a consequence of Theorem 2.1). This makes us to conclude that the BLUE of σ based on order statistics of a sample of size n arising from half-triangular distribution as defined by the pdf (3.3) and its variance is available, then the BLUE of σ based on the AOS of a sample of size n arising from the triangular distribution as defined by the pdf (1.1) and its variance can be obtained without any direct evaluation of means, variances and covariances of those AOS.

The moment relations for order statistics arising from general distributions which are symmetrically distributed about zero such as those described in Arnold et al. (1992), David and Nagaraja (2003) and Thomas and Samuel (1996) are useful for the easy evaluation of moments of order statistics from symmetric distributions, which can also apply to triangular distribution as such. However, application of Theorem 2.1 helps us to deal only with the evaluation of moments of order statistics arising from half-triangular distribution for using those values to the development of inference procedures on the scale parameter σ of triangular distribution. Moments of order statistics arising from standard half-triangular distribution is not seen discussed in the available literature. Hence, we have used Mathematica software to evaluate the means, variances and covariances of all order statistics $Y_{1:n}, Y_{2:n}, ..., Y_{n:n}$ for $n = 2(1)20$ arising from the standard half-triangular distribution defined by the pdf (3.3). Using those values we have determined the coefficients $c_{i,n}$ of the AOS $X_{(i:n)}$ in the estimate $\hat{\sigma}$ of σ as given in (3.10) for $i = 1, 2, ..., n; n = 2(1)20$ and those values are presented in Table 1. We have obtained further $\sigma^{-2}Var(\hat{\sigma})$ for $n = 2(1)20$ and those values are given in Table 2. Samuel and Thomas (2003) obtained the BLUE σ^* of σ based on order statistics of a random sample of size n from the triangular distribution with pdf (1.1). We have tabulated Var(σ^*) for $n = 2(1)20$ and is given in Table 2 with an objective of comparing the performance of the new estimator $\hat{\sigma}$ proposed in this paper with the already available estimator σ^*.

Table 1: Coefficient $c_{i,n}$ of $X_{(i:n)}$ involved in $\hat{\sigma} = \sum_{i=1}^{n} c_{i,n} X_{(i:n)}$ as an estimate of σ of triangular distribution for n=2(1)20.

n	**Coefficient $c_{i,n}$ of $X_{(i:n)}$ involved in $\hat{\sigma} = \sum_{i=1}^{n} c_{i,n} X_{(i:n)}$**																			
	$X_{(1:n)}$	$X_{(2:n)}$	$X_{(3:n)}$	$X_{(4:n)}$	$X_{(5:n)}$	$X_{(6:n)}$	$X_{(7:n)}$	$X_{(8:n)}$	$X_{(9:n)}$	$X_{(10:n)}$	$X_{(11:n)}$	$X_{(12:n)}$	$X_{(13:n)}$	$X_{(14:n)}$	$X_{(15:n)}$	$X_{(16:n)}$	$X_{(17:n)}$	$X_{(18:n)}$	$X_{(19:n)}$	$X_{(20:n)}$
2	0.3333	2.0000																		
3	0.1818	0.2727	1.6364																	
4	0.1200	0.1600	0.2400	1.4400																
5	0.0876	0.1095	0.1460	0.2190	1.3139															
6	0.0681	0.0816	0.1021	0.1361	0.204	1.2245														
7	0.0551	0.0643	0.0771	0.0964	0.1286	0.1928	1.157													
8	0.0459	0.0526	0.0613	0.0736	0.0919	0.1227	0.1839	1.1038												
9	0.0393	0.0442	0.0505	0.0589	0.0706	0.0884	0.1178	0.1767	1.0605											
10	0.0341	0.0379	0.0427	0.0488	0.0569	0.0683	0.0854	0.1138	0.1707	1.0243										
11	0.0301	0.0331	0.0367	0.0414	0.0473	0.0552	0.0662	0.0828	0.1104	0.1656	0.9934									
12	0.0269	0.0293	0.0322	0.0358	0.0403	0.0461	0.0537	0.0645	0.0806	0.1074	0.1612	0.9667								
13	0.0241	0.0264	0.0285	0.0315	0.0349	0.0393	0.0449	0.0524	0.0629	0.0786	0.1048	0.1572	0.9434							
14	0.0219	0.0237	0.0256	0.0279	0.0307	0.0342	0.0384	0.0439	0.0513	0.0615	0.0769	0.1025	0.1538	0.9226						
15	0.0201	0.0215	0.0232	0.0251	0.0274	0.0301	0.0335	0.0377	0.043	0.0503	0.0603	0.0755	0.1005	0.1507	0.9041					
16	0.0185	0.0197	0.0211	0.0228	0.0247	0.0269	0.0296	0.0329	0.0368	0.0424	0.0493	0.0592	0.0739	0.0986	0.1479	0.8874				
17	0.0171	0.0182	0.0194	0.0208	0.0224	0.0243	0.0264	0.0291	0.0323	0.0363	0.0415	0.0485	0.0581	0.0727	0.0969	0.1454	0.8722			
18	0.0159	0.0169	0.0179	0.0191	0.0204	0.022	0.0239	0.026	0.0286	0.0318	0.0358	0.0409	0.0477	0.0572	0.0715	0.0954	0.1431	0.8584		
19	0.0148	0.0157	0.0166	0.0175	0.0189	0.0201	0.0217	0.0235	0.0256	0.0282	0.0314	0.0352	0.0403	0.0469	0.0564	0.0705	0.0939	0.1409	0.8456	
20	0.0138	0.0148	0.0154	0.0164	0.0174	0.0185	0.0199	0.0195	0.0271	0.0234	0.0279	0.0308	0.0347	0.0397	0.0463	0.0556	0.0695	0.0927	0.1389	0.8339

Clearly, as $\hat{\sigma}$ depends on the observations which are the components of minimal sufficient statistic, the estimate $\hat{\sigma}$ is likely to be more efficient than the estimate σ^*. Thus our proposed new estimator $\hat{\sigma}$ of σ based on AOS is more preferable. To compare the performance of our estimator $\hat{\sigma}$, the relative efficiency $e(\hat{\sigma}/\sigma^*)$ is defined as $e(\hat{\sigma}/\sigma^*) = \frac{Var(\sigma^*)}{Var(\hat{\sigma})}$. The relative efficiencies for $n = 2(1)20$ are calculated and are presented in Table 2.From Table 2, it may be concluded that the estimate $\hat{\sigma}$ based on AOS is remarkably better than σ^* based on order statistics. The gain in efficiency observed in $\hat{\sigma}$ when compared with σ^* ranges from 25% to 138%.

4 ESTIMATION OF THE SCALE PARAMETER FROM CENSORED SAMPLES

When an outlier occurs in a sample drawn from a distribution which is symmetric about zero, it must be far off from zero either in the positive side or negative side. It is quite curious to know that unlike order statistics of the data, AOS capture that far off observation uniquely as the largest absolved order statistic $X_{(n:n)}$. But while working with order statistics this uniqueness is not materialized, as the farthest observation from zero may be either the smallest order statistic $X_{1:n}$ or the largest order statistic $X_{n:n}$. So to eliminate the effect of a suspected outlier from the sample with ordered data, theoretically a double censoring with one observation in the left ($X_{1:n}$) and one observation in the right ($X_{n:n}$) is required. Though for the triangular distribution also the above descriptions applies if we suspect more than two outliers in the data, then we have to generalize the estimate of σ by modifying the censoring scheme appropriately. In particular, if we have the reason to believe that there are k outlying observations in the data, then we go for right censoring of k of the AOS (those corresponding to the k observations in the data which lie most distantly from zero than others) and then estimate σ by using the available AOS $X_{(1:n)}, X_{(2:n)}, ..., X_{(n-k:n)}$ where k is any positive integer such that $1 \leq k \leq n-2$. The estimation procedure for σ in the censoring scheme then follows from the theorem given below.

Table 2: Variances of (i) BLUE $\hat{\sigma}$ based on AOS (ii) BLUE σ^* based on order statistics of the scale parameter σ of triangular distribution and the relative efficiency : $e(\hat{\sigma}/\sigma^*)$ for n=2(1)20.

n	$\sigma^{-2}Var(\hat{\sigma})$	$\sigma^{-2}Var(\sigma^*)$	$e(\hat{\sigma}/\sigma^*)$
2	0.2222	0.5306	2.3879
3	0.1364	0.2415	1.7705
4	0.0960	0.1511	1.5739
5	0.0730	0.1080	1.4795
6	0.0583	0.0830	1.4237
7	0.0482	0.0668	1.3859
8	0.0409	0.0555	1.3570
9	0.0354	0.0473	1.3362
10	0.0311	0.0409	1.3151
11	0.0276	0.0361	1.3080
12	0.0248	0.0321	1.2944
13	0.0225	0.0289	1.2845
14	0.0205	0.0262	1.2781
15	0.0188	0.0239	1.2713
16	0.0173	0.0220	1.2644
17	0.0162	0.0203	1.2531
18	0.0151	0.0188	1.2450
19	0.0141	0.0176	1.2482
20	0.0132	0.0165	1.2500

Theorem 4.1. *Suppose* $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ *are the AOS of a random sample of size n drawn from the triangular distribution with pdf defined in (1.1). Let* $Y_{1:n}, Y_{2:n}, ..., Y_{n:n}$ *be the order statistics of a random sample of size n drawn from the standard half-triangular distribution given in (3.3). Suppose k observations corresponding to those with largest k absolute values are censored so that the vector of the remaining AOS is* $\underset{\sim}{X}_{n-k} = (X_{(1:n)}, X_{(2:n)}, ..., X_{(n-k:n)})'$. *Define* $\underset{\sim}{Y}_{n-k} = (Y_{1:n}, Y_{2:n}, ..., Y_{n-k:n})'$, $E(\underset{\sim}{Y}_{n-k}) = \underset{\sim}{\alpha}_{n-k} = (\alpha_{1:n}, \alpha_{2:n}, ..., \alpha_{n-k:n})'$ *and let the dispersion matrix of* $\underset{\sim}{Y}_{n-k}$ *is denoted by* A_{n-k}. *In this case, we write* $E(\underset{\sim}{X}_{n-k}) = \underset{\sim}{\alpha}_{n-k}\sigma$, $D(\underset{\sim}{X}_{n-k}) = A_{n-k}\sigma^2$. *Then the BLUE* $\hat{\sigma}_{k,n}$ *of* σ *based on the censored AOS is given by*

$$\hat{\sigma}_{k,n} = (\underset{\sim}{\alpha}'_{n-k}\mathbf{A}^{-1}_{n-k}\underset{\sim}{\alpha}_{n-k})^{-1}\underset{\sim}{\alpha}'_{n-k}\mathbf{A}^{-1}_{n-k}\underset{\sim}{X}_{n-k}. \tag{4.1}$$

The variance of $\hat{\sigma}_{k,n}$ *is given by*

$$Var(\hat{\sigma}_{k,n}) = (\underset{\sim}{\alpha}'_{n-k}\mathbf{A}^{-1}_{n-k}\underset{\sim}{\alpha}_{n-k})^{-1}\sigma^2. \tag{4.2}$$

Proof. The proof of the above theorem follows easily by the application of Gauss-Markov theorem. □

Remark 4.1. *One can write (4.1) as a linear function of* $X_{(1:n)}, X_{(2:n)}, ..., X_{(n-k:n)}$ *as*

$$\hat{\sigma}_{k,n} = \sum_{i=1}^{n-k} c^{(k)}_{i,n} X_{(i:n)}, \tag{4.3}$$

where $c^{(k)}_{i,n}$, $i = 1, 2, ..., n-k$ *are appropriate constants.*

Table 3: Coefficient $c_{i,10}^{(k)}$ of $X_{(i:10)}$ involved in $\hat{\sigma}_{k,10} = \sum_{i=1}^{10-k} c_{i,10}^{(k)} X_{(i:10)}$, $Var(\hat{\sigma}_{k,10})$ and efficiency $e(\hat{\sigma}_{k,10}/\hat{\sigma})$ of $\hat{\sigma}_{k,10}$ relative to $\hat{\sigma}$ for the scale parameter σ of the triangular distribution.

Statistic used	**Coefficient $c_{i,10}^{(k)}$ of $X_{(i:10)}$ involved in $\hat{\sigma}_{k,10} = \sum_{i=1}^{10-k} c_{i,10}^{(k)} X_{(i:10)}$ given along columns.**								
$X_{(i:10)}$	$\hat{\sigma}$ $(k=0)$	$\hat{\sigma}_{1,10}$ $(k=1)$	$\hat{\sigma}_{2,10}$ $(k=2)$	$\hat{\sigma}_{3,10}$ $(k=3)$	$\hat{\sigma}_{4,10}$ $(k=4)$	$\hat{\sigma}_{5,10}$ $(k=5)$	$\hat{\sigma}_{6,10}$ $(k=6)$	$\hat{\sigma}_{7,10}$ $(k=7)$	$\hat{\sigma}_{8,10}$ $(k=8)$
$X_{(1:10)}$	0.0341	0.0518	0.0699	0.0913	0.1183	0.1549	0.2088	0.2975	0.4737
$X_{(2:10)}$	0.0379	0.0576	0.0778	0.1014	0.1314	0.1721	0.2320	0.3306	10.000
$X_{(3:10)}$	0.0427	0.0648	0.0875	0.1141	0.1478	0.1936	0.2610	6.3223	
$X_{(4:10)}$	0.0488	0.0741	0.0999	0.1304	0.1689	0.2213	4.4739		
$X_{(5:10)}$	0.0569	0.0864	0.1166	0.1521	0.1971	3.3559			
$X_{(6:10)}$	0.0683	0.1037	0.1399	0.1826	2.6016				
$X_{(7:10)}$	0.0854	0.1296	0.1749	2.0536					
$X_{(8:10)}$	0.1138	0.1728	1.6329						
$X_{(9:10)}$	0.1707	1.2960							
$X_{(10:10)}$	1.0243								
$\sigma^{-2}Var(\hat{\sigma}_{k,n})$	0.0310	0.0471	0.0636	0.0830	0.1075	0.1408	0.1898	0.2705	0.4306
$e(\hat{\sigma}_{k,n}/\hat{\sigma})$	-	0.6586	0.4879	0.3741	0.2887	0.2204	0.1635	0.1148	0.0721

The method of estimation of σ by $\hat{\sigma}_{k,n}$ as described in Theorem 4.1 using censored AOS arising from the triangular distribution is attempted for a sample of size 10 and for each of $k = 1, 2, 3, ..., 8$. For each $k = 1, 2, 3, ..., 8$, we have computed the numerical value of the coefficients $c_{i,10}^{(k)}$ of $X_{(i:10)}$ involved in $\hat{\sigma}_{k,10}$ for $i = 1, 2, ..., 10-k$ and $\sigma^{-2}Var(\hat{\sigma}_{k,10})$ and these computed values are given in Table 3. The relative efficiency $e(\hat{\sigma}_{k,10}/\hat{\sigma}) = \frac{Var(\hat{\sigma})}{Var(\hat{\sigma}_{k,10})}$ of $\hat{\sigma}_{k,10}$ when compared with the BLUE $\hat{\sigma}$ as derived in (3.10) is again computed for each $k = 1, 2, 3, ..., 8$ and the computed values are also presented in Table 3. Note that when $k = 0$, the estimate $\hat{\sigma}_{0,10}$ is same as $\hat{\sigma}$ as given in (3.10) for $n = 10$.

From Table 3, we observed that initially for $k = 1$, there is not much reduction noticed on the relative efficiency keeping the relative efficiency more than 65%. However the reduction noticed in the relative efficiencies become little more but changes at a sluggish rate when more number of extreme absolved order statistics are censored. For example, if $n = 10$ and $k = 8$ then in the estimator $\hat{\sigma}_{8,10}$ altogether eight AOS are censored and hence it utilizes only two AOS, $X_{(1:10)}$ and $X_{(2:10)}$ for estimating σ. From these two AOS, the efficiency observed in $\sigma_{\hat{8,10}}$ relative to the estimator $\hat{\sigma}$ (in which 8 out of 10 AOS are involved) is more than 7%. This makes to comment that AOS based estimators for the scale parameter σ of triangular distribution appears to be robust.

5 U-STATISTICS AS ESTIMATOR FOR THE SCALE PARAMETER

Sreekumar and Thomas (2007) have introduced the technique of constructing U-statistic estimators for the location and scale parameters of a distribution by taking BLUE based on order statistics of those parameters as kernels. Sreekumar and Thomas (2007) have also narrated a technique of obtaining the variance of those U-statistics and illustrated their method to estimate the parameters of log-gamma distribution. For applications of U-statistics based on BLUE's based on order statistic in estimating the location and scale parameters of distributions see also, Sreekumar and Thomas (2008), Thomas and Baiju (2012, 2015), Thomas and Priya (2015, 2016), Sreekumar and Thomas (2008). If $\mathscr{F}_1$ is the family of distributions which are all symmetrically distributed about zero, Thomas and Anjana (2021) explained that in order to deal with the estimation of σ based on AOS arising from $f(x, \sigma) \in \mathscr{F}_\theta^{(1)}$, it is enough to deal with the estimation of σ based on order statistics arising from the folded distribution with pdf $g(z, \sigma) = \frac{2}{\sigma} f_0(\frac{z}{\sigma}), 0 < z < \infty$. Then to obtain the estimate of σ based on AOS arising out of a random sample of size n from $f(x, \sigma)$, we attempt for the corresponding estimate of σ based on order statistics of a random sample of size n arising from the pdf $g(z, \sigma)$ and replace in it each order statistic by the correspondingly ordered AOS. The variance expression for the estimates remains the same for both cases. As a result of these logical arguments, Thomas and Anjana (2021) derived a U-statistic using BLUE based on AOS as kernel for σ involved in $f(x, \sigma)$ and also showed that it is enough to derive a U-statistic using BLUE based on order statistics as kernel for σ involved in $g(z, \sigma)$ and finally replace each order statistic in the estimate by the correspondingly ordered AOS. Now the method of constructing the U-statistic for σ involved in triangular distribution is described here using a BLUE based on absolved order statistics.

Let $X_1, X_2, ..., X_n$ be a random sample of size n drawn from triangular distribution, defined in (1.1) so that the absolute values of the observations $|X_1|, |X_2|, ..., |X_n|$ may be considered as a random sample of size n drawn from half-triangular distribution defined in (3.2). Let $Z_1, Z_2, ..., Z_m$ be an initial random sample of size m drawn from the half- triangular distribution defined in (3.2). Let $\underset{\sim}{Z} = (Z_{1:m}, Z_{2:m}, ..., Z_{m:m})'$ be the vector of order statistics obtained from $Z_1, Z_2, ..., Z_m$. If $\underset{\sim}{V} = (V_{1:m}, V_{2:m}, ..., V_{m:m})'$ is considered as the vector of order statistics of a random sample of size m drawn from the pdf given by (3.3), then the distribution of these order statistics are independent of σ.

Let $E(\underset{\sim}{V}) = \underset{\sim}{\beta} = (\beta_{1:m}, \beta_{2:m}, ..., \beta_{m:m})'$ be the vector of expectation of $\underset{\sim}{V}$ and let the dispersion matrix of $\underset{\sim}{V}$ be denoted by $\mathbf{G}$. Then from David and Nagaraja (2003), the BLUE of σ involved in (3.2) based on order statistics is given by

$$\hat{\sigma} = (\underset{\sim}{\beta}'\mathbf{G^{-1}}\underset{\sim}{\beta})^{-1}\underset{\sim}{\beta}'\mathbf{G^{-1}}\underset{\sim}{Z} \tag{5.1}$$

Table 4: Variances components $\xi_c^{(m)}$ for $c = 1(1)m$ and $m = 2(1)5$ for triangular distribution.

m	c	$\sigma^{-2}\xi_c^{(m)}$	m	c	$\sigma^{-2}\xi_c^{(m)}$	m	c	$\sigma^{-2}\xi_c^{(m)}$
2	1	0.1009	4	1	0.0209	5	1	0.0125
	2	0.2222		2	0.0435		2	0.0260
3	1	0.0402		3	0.0683		3	0.0403
	2	0.0851		4	0.0960		4	0.0559
	3	0.1364					5	0.0730

and variance of $\hat{\sigma}$ is given by

$$Var(\hat{\sigma}) = (\underset{\sim}{\beta}'\mathbf{G^{-1}}\underset{\sim}{\beta})^{-1}\sigma^2. \tag{5.2}$$

We may write $\hat{\sigma}$ also as

$$h(Z_1, Z_2, ..., Z_m) = c_{1:m}Z_{1:m} + c_{2:m}Z_{2:m} + ... + c_{m:m}Z_{m:m}, \tag{5.3}$$

where $c_{1:m}, c_{2:m}, ..., c_{m:m}$ are constants which are determined from (5.1). Now from Sreekumar and Thomas (2007), we can easily write the U-statistic for a random sample $Z_1, Z_2, ..., Z_n$ $(n > m)$ drawn from the distribution with pdf (3.2) based on the kernel (5.3) as

$$U_n^{(m)} = \frac{1}{\binom{n}{m}} \sum_{r=1}^{n} \left[\sum_{j=1}^{m} \binom{n-r}{m-j} \binom{r-1}{j-1} c_{j:m} \right] Z_{r:n}, \tag{5.4}$$

where we define for non-negative integers p and q, $\binom{p}{q} = 0$ for $p < q$. If we write

$$\xi_c^{(m)} = Cov[h(Z_1, Z_2, ..., Z_c, Z_{c+1}, ..., Z_m), h(Z_1, Z_2, ..., Z_c, Z_{m+1}, ..., Z_{2m-c})], \tag{5.5}$$

for, $c = 1, 2, ..., m$, then the $Var(U_n^{(m)})$ is given by

$$Var[U_n^{(m)}] = \frac{1}{\binom{n}{m}} \sum_{c=1}^{m} \binom{m}{c} \binom{n-m}{m-c} \xi_c^{(m)}. \tag{5.6}$$

Clearly $\xi_m^{(m)} = Var(h(Z_1, Z_2, ..., Z_m)$ and is given by (5.2).

The components $\xi_c^{(m)}$ for $c = 1, 2, ..., m-1$ involved is $Var[U_n^{(m)}]$ though look very simple, obtaining those values exactly is somewhat computationally difficult with respect to a given kernel. However Thomas and Anjana (2021) have given a methodology to obtain those values and the version of that methodology to the present problem is given below.

If we put $n = m + k$ in (5.6), then we obtain

$$Var(U_{m+k}^{(m)}) = \frac{1}{\binom{m+k}{m}} \left\{ \binom{m}{m-k}\binom{k}{k}\xi_{m-k}^{(m)} + \binom{m}{m-k+1}\binom{k}{k-1}\xi_{m-k+1}^{(m)} + ... + \binom{m}{m}\binom{k}{0}\xi_m^{(m)} \right\}, \tag{5.7}$$

for, $k = 1, 2, ..., m-1$.

On putting $n = m + k$ in (5.4) we obtain

$$U_{m+k}^{(m)} = \frac{1}{\binom{m+k}{m}} \left\{ \left[\sum_{j=1}^{m} \binom{m-k-1}{m-j}\binom{0}{j-1} c_{j:m} \right] Z_{1:m+k} + \left[\sum_{j=1}^{m} \binom{m+k-2}{m-j}\binom{1}{j-1} c_{j:m} \right] Z_{2:m+k} + ... + \left[\sum_{j=1}^{m} \binom{0}{m-j}\binom{m+k-1}{j-1} c_{j:m} \right] Z_{m+k:m+k} \right\}. \tag{5.8}$$

We can write the above equation also as

$$U_{m+k}^{(m)} = \underset{\sim}{\mathbf{b}}_{\mathbf{m+k}}^{'} \underset{\sim}{\mathbf{Z}}_{\mathbf{m+k}}, \tag{5.9}$$

where $\underset{\sim}{\mathbf{Z}}_{\mathbf{m+k}} = (Z_{1:m+k}, Z_{2:m+k}, ..., Z_{m+k:m+k})'$ and

$$\binom{m+k}{m} \underset{\sim}{\mathbf{b}}_{\mathbf{m+k}}^{'} = \left(\sum_{j=1}^{m} \binom{m+k-1}{m-j}\binom{0}{j-1} c_{j:m}, \sum_{j=1}^{m} \binom{m+k-2}{m-j}\binom{1}{j-1} c_{j:m}, ..., \sum_{j=1}^{m} \binom{0}{m-j}\binom{m+k-1}{j-1} c_{j:m} \right), \tag{5.10}$$

where in both (5.8) and (5.10) we define $\binom{p}{q} = 0$ for $p < q$. From (5.9) we write

$$Var(U_{m+k}^{(m)}) = (\underset{\sim}{\mathbf{b}}_{\mathbf{m+k}}{}^{'} \mathbf{G}_{\mathbf{m+k}} \underset{\sim}{\mathbf{b}}_{\mathbf{m+k}})\sigma^2, k = 1, 2, ..., m-1, \tag{5.11}$$

where $\mathbf{G_{m+k}}$ is the variance-covariance matrix of the vector of order statistics of a random sample of size m+k drawn from the half-triangular distribution with pdf (3.3). Now from (5.7) and (5.11) we write

$$\binom{m}{m-k}\binom{k}{k}\xi_{m-k}^{(m)} + \binom{m}{m-k+1}\binom{k}{k-1}\xi_{m-k+1}^{(m)} + ... + \binom{m}{m-1}\binom{k}{1}\xi_{m-1}^{(m)} = \binom{m+k}{m}(\underset{\sim}{\mathbf{b_{m+k}}}'\mathbf{G_{m+k}}\underset{\sim}{\mathbf{b_{m+k}}})\sigma^2 - \xi_m^{(m)} \tag{5.12}$$

for, $k = 1, 2, ..., m-1$, where we define for non-negative integers p and q, $\binom{p}{q} = 0$ for $p < q$. The above system of equations can be written by the following matrix equation:

$$\begin{bmatrix} 0 & 0 & ... & 0 & \binom{m}{m-1}\binom{1}{1} \\ 0 & 0 & ... & \binom{m}{m-2}\binom{2}{2} & \binom{m}{m-1}\binom{2}{1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \binom{m}{1}\binom{m-1}{m-1} & \binom{m}{2}\binom{m-1}{m-2} & ... & \binom{m}{m-2}\binom{m-1}{2} & \binom{m}{m-1}\binom{m-1}{1} \end{bmatrix} \begin{bmatrix} \xi_1^{(m)} \\ \xi_2^{(m)} \\ \vdots \\ \xi_{m-1}^{(m)} \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{m-1} \end{bmatrix}, \tag{5.13}$$

where $\omega_k = \binom{m+k}{m}(\underset{\sim}{\mathbf{b_{m+k}}}'\mathbf{G_{m+k}}\underset{\sim}{\mathbf{b_{m+k}}})\sigma^2 - \xi_m^{(m)}, k = 1, 2, ..., m-1.$

If we write H to denote the coefficient matrix of the left side of (5.13) and $\underset{\sim}{\omega}$ to denote the vector in the right side of (5.13), then we have

$$(\xi_1^{(m)}, \xi_2^{(m)}, \ldots, \xi_{m-1}^{(m)})' = H^{-1}\underset{\sim}{\omega}.$$

The values of $\xi_1^{(m)}, \xi_2^{(m)}, \ldots, \xi_{m-1}^{(m)}$ can be obtained from the above equation and together with them if use $\xi_m^{(m)}$ as given in (5.2) and (5.6), we obtain the variance of the U-statistic $U_n^{(m)}$.

Thus as a result of the theory developed in this paper, we write the U-statistic using the BLUE of σ based on AOS as a kernel of degree m as $U_{n,A}^{(m)}$ which we trace out from (5.4) and is given by

$$U_{n,A}^{(m)} = \frac{1}{\binom{n}{m}} \sum_{r=1}^{n} \left[\sum_{j=1}^{m} \binom{n-r}{m-j}\binom{r-1}{j-1} c_{j:m} \right] X_{(r:n)}, \tag{5.14}$$

where, $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ are the AOS of a random sample of size n drawn from the triangular distribution with pdf (1.1). The variance of $U_{n,A}^{(m)}$ is given by

$$Var[U_{n,A}^{(m)}] = \frac{1}{\binom{n}{m}} \sum_{c=1}^{m} \binom{m}{c} \binom{n-m}{m-c} \xi_c^{(m)}, \tag{5.15}$$

where $\xi_c^{(m)}, c = 1, 2, ..., m-1$ are solved from (5.13) and $\xi_m^{(m)}$ (the variance of the kernel) is given as in (5.2). It is to be noted that provided $c_{j:m}$ for $j = 1, 2, ..., m$ are determined then for any sample size n however large may be, one can obtain explicitly an unbiased, strongly consistent and asymptotically normal type of estimator $U_{n,A}^{(m)}$ for σ and is as given in (5.14). Similarly, if the means, variances and covariances of order statistics arising from half-triangular distribution for all sample sizes from m to $2m-1$ are obtained, then one can determine the exact variance of the estimator $U_{n,A}^{(m)}$ for any sample size however large it may be.

Now, in order to find that the minimal sufficient property of AOS arising from the triangular distribution with pdf (1.1) has induced any improvement on the U-statistic $U_{n,A}^{(m)}$, we obtain the variance $Var(U_{n,O}^{(m)})$ of a similar U-statistic $U_{n,O}^{(m)}$ constructed from the BLUE based on order statistics as kernel of the same degree arising from (1.1). If $Var(U_{n,A}^{(m)}) < Var(U_{n,O}^{(m)})$, then as usual we claim that $U_{n,A}^{(m)}$ is a better estimate of σ than the estimate $U_{n,O}^{(m)}$. Then the efficiency of $U_{n,A}^{(m)}$ relative to $U_{n,O}^{(m)}$ is denoted by $e^{(m)}$ and is defined as $e^{(m)} = \frac{Var(U_{n,O}^{(m)})}{Var(U_{n,A}^{(m)})}$.

To illustrate the theory discussed above to estimate the scale parameter σ in (1.1), we consider BLUE's based on AOS with initial sample sizes m = 2, 3, 4 and 5 drawn from the triangular distribution and take them as kernels of degrees 2, 3, 4 and 5 respectively. Using the estimate given by (3.8) and its linear representation given in (3.10), kernels in terms of order statistics arising from half-triangular distribution are derived and are given by

$$h(Z_1, Z_2) = 0.3333 Z_{1:2} + 2.000 Z_{2:2},$$

$$h(Z_1, Z_2, Z_3) = 0.1818 Z_{1:3} + 0.2727 Z_{2:3} + 1.6364 Z_{3:3},$$

Table 5: Variances of U-statistics $U_{n,A}^{(m)}$, $U_{n,O}^{(m)}$ from triangular distribution and relative efficiencies $e^{(m)} = \frac{Var(U_{n,O}^{(m)})}{Var(U_{n,A}^{(m)})}$ for different n and m.

Sample Size	Variances of U-statistics								Relative efficiencies $e^{(m)}$			
	$U_{n,A}^{(2)}$	$U_{n,O}^{(2)}$	$U_{n,A}^{(3)}$	$U_{n,O}^{(3)}$	$U_{n,A}^{(4)}$	$U_{n,O}^{(4)}$	$U_{n,A}^{(5)}$	$U_{n,O}^{(5)}$	$e^{(2)}$	$e^{(3)}$	$e^{(4)}$	$e^{(5)}$
10	0.0408	0.047	0.0371	0.0463	0.0349	0.0451	0.0334	0.0438	1.1520	1.2479	1.2923	1.3114
15	0.0271	0.0295	0.0245	0.0291	0.0229	0.0283	0.0218	0.0274	1.0886	1.1878	1.2358	1.2569
20	0.0203	0.0215	0.0183	0.0212	0.0171	0.0206	0.0161	0.0199	1.0591	1.1585	1.2047	1.2360
30	0.0135	0.0140	0.0122	0.0137	0.0113	0.0133	0.0107	0.0129	1.0370	1.1229	1.1769	1.2056
40	0.0101	0.01022	0.0091	0.0101	0.0085	0.0098	0.0079	0.0095	1.0189	1.1099	1.1529	1.2025
60	0.0067	0.0068	0.0061	0.0067	0.0057	0.0065	0.0053	0.0062	1.0149	1.0984	1.1404	1.1698
80	0.00503	0.0051	0.0045	0.0049	0.00425	0.0048	0.0041	0.0046	1.0139	1.0889	1.1429	1.1795
100	0.004	0.0041	0.0036	0.0039	0.0034	0.0038	0.0032	0.0037	1.0250	1.0833	1.1176	1.1563

$$h(Z_1, Z_2, Z_3, Z_4) = 0.1200Z_{1:4} + 0.1600Z_{2:4} + 0.2400Z_{3:4} + 1.4400Z_{4:4},$$

$$h(Z_1, Z_2, Z_3, Z_4, Z_5) = 0.0876Z_{1:5} + 0.1095Z_{2:5} + 0.146Z_{3:5} + 0.219Z_{4:5} + 1.3139Z_{5:5}.$$

Then using the above coefficients of the order statistics arising from half-triangular distribution in (5.14), the respective U-statistics $U_{n,A}^{(2)}$, $U_{n,A}^{(3)}$, $U_{n,A}^{(4)}$ and $U_{n,A}^{(5)}$ are obtained. Further, the components of variances $\xi_c^{(m)}$ for $c = 1, 2, ..., m-1$, $m = 2, 3, 4$ *and* 5 are computed and those values are tabulated in Table 4.

Using those values of $\xi_c^{(m)}$ for each of m= 2, 3, 4, 5 Var($U_{n,A}^{(m)}$) for $n = 10(5)20(10)$ 40(20)100 are determined and are given in Table 5. As described in Thomas and Sreekumar (2008), Var($U_{n,O}^{(m)}$) and efficiency $e^{(m)}$ of $U_{n,A}^{(m)}$ relative to $U_{n,O}^{(m)}$ for $n =$ 10(5)20(10)40(20)100 are also calculated and these values are also presented in Table 5.

From Table 5, it can be seen that all efficiencies of U-statistics $U_{n,A}^{(m)}$ generated from BLUE based on AOS as kernels relative to the U-statistics $U_{n,O}^{(m)}$ generated from BLUE based on classical order statistics as kernels are greater than unity, on estimating the scale parameter of triangular distribution.

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