Journal of the Kerala Statistical Association
Vol.32, December 2021, 40-49 *ISSN 2249-4553*

Some Properties of Laplace Process

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ABSTRACT

Laplace process was introduced in Satheesh (1990) as a possible alternative to Wiener process. Here we discuss some properties of this process *viz.* correlation, martingales, finite dimensional distributions and sample paths. A review of the rich theoretical developments over Laplace process, driven by data, over the years is also given.

Key words and Phrases: *Fractional process, geometric Brownian motion, long memory, Laplace, Lévy, Ornstein-Uhlenbeck, Rosenblatt and stable processes.*

1 Introduction

There are situations when a Laplace model is preferred to a Gausian one. While Hsu (1979) used it to model the pooled position errors in a large navigation system, Sethia and Anderson (1984) used a stationary autoregressive model with Laplace marginals in communication engineering. Such possibilities motivated the introduction of Laplace process in Satheesh (1990). There some martingales of interest were discussed and it was shown that a Laplace process is subordinated to a standard Wiener process by a directing gamma process.

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Definition 1.1. *A random process* $\{X(t), t \geq 0\}$ *with stationary and independent increments is a Laplace process if* $X(0) = 0$ *and the characteristic function (CF) of* $X(1)$ *is* $\phi(s, 1) = 1/(1 + s^2)$.

Looking back, one finds that rich theoretical advances driven by empirical evidence have taken place over the last 30 years. Hence, before recording our results here, we sketch these developments.

Modelling stock prices with stochastic processes began with Bachelier (1900) when he modelled logarithm of prices as a Brownian motion (BM) with drift. One of the limitations of this model was that it failed to account for negative prices. Samuelson (1965) suggested the use of geometric BM (GBM) which was used by Black and Scholes (1973) and Merton (1973) to demonstrate the fluctuations in European markets. Though Black-Scholes (BS) model won the 1997 Nobel prize for Economics, it was rejected on empirical grounds, unable to capture important features of data in the market, see Barndorff-Nielsen and Shephard (2001).

Madan and Seneta (1990) proposed the variance gamma (VG) processes (same as the Laplace process studied in this paper) to model long taildness and closure of increments under summation inherent in data. The idea here is that a larger class of Lévy processes can be derived from the BM $\{B(t)\}$ by randomizing its time parameter t using a positive continuous random variable T. The advantage is, while certain features of $\{B(t)\}$ are retained, those of T can be augmented to alter some others to get new useful processes. See, e.g. Schoutens (2003) for more on Lévy processes in the context.

One of the first specific Lévy models used in the context of stock-price modelling is the VG model by Madan and Seneta (1990). As the name suggests, variance of a normal law is randomised using a gamma law, or in terms of processes, the time parameter of BM is randomized using a gamma process; the result is Laplace process, studied in Satheesh (1990). It is also called normal-gamma process by Bibby and Sorensen (2003). A three parameter extension of VG model, that can be tuned for the observed volatility, skewness and kurtosis has been introduced by Madan *et al.* (1998). Kotz *et al.* (2001) names it as Laplace motion. Typically, Laplace motion accounts for distributions of increments that are more peaked at the mode with thick tails. Though Lévy models, in general, were a significant improvement over the BS model, they still fell short in accounting for stochastic volatility (SV).

Barndorff-Nielsen and Shephard (2001) brought in Ornstein-Uhlenbeck (OU) processes to model SV. They observed that every assumption underlying BS model was empirically rejected. OU models offer plenty of analytic tractability, the possibility of capturing important distributional deviations from Gaussianity and flexible modelling of dependence structures. They constructed continuous time SV processes as superposition of positive OU processes.

While Lévy processes, based on infinitely divisible laws, have been successfully used to capture skewness and kurtosis observed in financial data, self-decomposable laws lead to stationary OU processes that capture SV. The need to model heavier or thicker tails, a feature of data on market prices, has lead to the use of non-Gaussian stable processes, see Samarodnitsky and Taqqu (1994) and Jondeau *et al.* (2007). Non-Gaussian stable models form a subclass of OU, which is a subclass of Lévy.

Many datasets on logarithm of price changes exhibit long-range dependence (LRD) or long memory, conceived by Mandelbrot and Wallis (1968). Since the feature of LRD can be modelled by the property of self-similarity, which a fractional BM (FBM) possesses, such datasets can be modelled using FBM, Mandelbrot and Ness (1968). Data on logarithms of hydraulic conductivity exhibited peakedness at the mode and thick tails in addition to LRD and fractional Laplace motion (FLM) was introduced by Kozubowski *et al.* (2006) as FBM failed to model these appropriately. This is subordination of FBM to gamma process, see also Gajda *et al.* (2017). Kumar *et al.* (2017) extended this by subordinating FBM to inverse gamma process. To account for thicker tails in the context, fractional Lévy stable motions (FLSM) with infinite variance were considered in Pipiras and Taqqu (2017) which were extended by subordination to a gamma process in Gajda *et al.* (2018).

Rosenblatt process is one of the "Hermite processes" which are limits of normalized sums of LRD random variables, Taqqu (2011). The simplest Hermite process is the FBM and the Rosenblatt process is the simplest non-Gaussian Hermite process.

It has the same covariance as the FBM, is self-similar and hence can be used to model LRD. Subordinating it to a gamma process we get the Rosenblatt Laplace motion as a potential alternative to FLSM, see Lupascu-Stamate and Tudor (2019).

A standout feature of the developments reviewed above is that it has been data driven. At various stages since Bachelier (1900), the models were found inadequate to capture certain features in the data and got updated to account for that. Though mainly used to model data from finance and hydraulic conductivity, these models are also applied to areas *viz.* condensed matter physics, economics, geophysics, hydrology and telecommunications, see Kotz *et al.* (2001). While Gaussian process is completely specified by its mean and auto-correlation function, non-Gaussian models discussed above command a much richer parametrization and thus offer additional merit of flexibility and variety.

Similar parallel developments are there for the non-Gaussian auto-regressive models. For an update on these see, Bouzar and Satheesh (2008), Bakouch *et al.* (2013), Balakrishna (2021) and also the books already referred to.

Now, let us record our results. Since for Laplace process $\{X(t), t \geq 0\}$, $E(X(t)) = 0$ and $V(X(t)) = E(X^2(t)) = 2t$, Satheesh (1990), we write $X(t) \sim \mathcal{L}(0, 2t)$. In the next section we discuss its correlation structure, certain other martingales of interest, finite dimensional distributions (FDD) and nature of sample paths and compare them with those of Wiener process/ BM.

2 Properties

Let $\{X(t), t \geq 0\}$ be Laplace process. Then for $s < t$,

$$\begin{aligned}
Cov[X(t), X(s)] &= E[X(t)X(s)] - E[X(t)]E[X(s)] \\
&= E[X(t)X(s)] = E[(X(t) - X(s) + X(s))X(s)] \\
&= E[(X(t) - X(s))X(s) + X^2(s)] \\
&= E[X(t) - X(s)]E(X(s)) + E(X^2(s)) \\
&= 0 + 2s = 2s,
\end{aligned}$$

since the increments are independent, $X(t) - X(s)$ is independent of $X(s)$ for $s < t$. For $s > t$ we obtain $Cov[X(t), X(s)] = 2t$, in a similar way. Hence, $Cov[X(t), X(s)] = 2\min\{s, t\} = 2(s \wedge t)$. Now the correlation coefficient between $X(s)$ and $X(t)$ is

$$\rho_{s,t} = \frac{2(s \wedge t)}{\sqrt{2t}\sqrt{2s}} = \sqrt{\frac{s}{t}}, \text{ if } s < t \text{ and } \sqrt{\frac{t}{s}}, \text{ if } s > t. \text{ Hence } \rho_{s,t} = \sqrt{\frac{s \wedge t}{s \vee t}}.$$

Note that $\rho_{s,t}$ is identical to that of Wiener process. Certain martingales of interest are discussed below.

Proposition 2.1. *For the Laplace process $\{X(t), t \geq 0\}$ on the probability space $(\Omega, \mathcal{F}, P)$, $X(t)$ and $X^2(t) - 2t$ are martingales.*

Proof. Note that $X(t)$ is $\mathcal{F}_t$ adapted and $X(t) - X(s)$ is independent of $\mathcal{F}_s,\ s < t$. Since $E(X(t)|\mathcal{F}_s) = E(X(t)-X(s)+X(s)|\mathcal{F}_s) = E(X(t)-X(s)|\mathcal{F}_s)+E(X(s)|\mathcal{F}_s) = X(s)$, $X(t)$ is a martingale. Again,

$$\begin{aligned} E\{X^2(t) - 2t|\mathcal{F}_s\} &= E\{(X(t) - X(s) + X(s))^2 - 2t|\mathcal{F}_s\} \\ &= E\{[X(t) - X(s)]^2 + 2[X(t) - X(s)]X(s) + X^2(s) - 2t|\mathcal{F}_s\} \\ &= 2(t - s) + 0 + X^2(s) - 2t \\ &= X^2(s) - 2s. \end{aligned}$$

Since $|X^2(t) - 2t| \leq X^2(t) + 2t$, $E(|X^2(t) - 2t|) \leq E(X^2(t) + 2t) = E(X^2(t)) + 2t = 4t < \infty$. Hence, $X^2(t) - 2t$ is a martingale. □

Note that for Wiener process, $\{W(t), t \geq 0\}$, $W^2(t) - t$ is a martingale, which is a sub-martingale for the Laplace process.

To derive the *FDD* of Laplace process we proceed as follows. The CF of the increment $X(t+p) - X(t), p > 0$ is $\phi(s, p) = (1 + s^2)^{-p} = (1 + is)^{-p}(1 - is)^{-p}$. This shows that $\phi(s, p)$ is the CF of the difference of two *i.i.d* gamma(p) variables. Let X and Y are *i.i.d* gamma(p) variables. The joint density of (X, Y) is

$$f(x, y) = \frac{1}{(\Gamma(p))^2} x^{p-1} y^{p-1} e^{-(x+y)};\ x, y > 0,\ p > 0.$$

The Jacobian of the transformation $U = X$ and $V = X - Y$ is unity and hence the joint density of (U, V) is

$$g(u,v) = \frac{1}{(\Gamma(p))^2} u^{p-1}(u-v)^{p-1}e^{-(2u-v)}, u > max\{0,v\}, v \in \mathbf{R}.$$

Since the distribution of V is symmetric about zero, the density of V for $v > 0$ is

$$g_1(v) = \frac{1}{(\Gamma(p))^2}\int_v^\infty u^{p-1}(u-v)^{p-1}e^{-(2u-v)}\ du,\ v > 0.$$

Setting $W = \frac{U}{V}$ we get,

$$\begin{aligned} g_1(v) &= \frac{e^v v^{2p-1}}{\Gamma(p)}\frac{1}{\Gamma(p)}\int_1^\infty w^{p-1}(w-1)^{p-1}e^{-2vw}\ dw \\ &= \frac{e^{-v}v^{2p-1}}{\Gamma(p)}\frac{e^{2v}}{\Gamma(p)}\int_1^\infty w^{p-1}(w-1)^{p-1}e^{-2vw}\ dw \\ &= \frac{e^{-v}v^{2p-1}}{\Gamma(p)}U(p,2p,2v), v > 0, \end{aligned}$$

using formula 13.2.6, on p.505 in Abramovitz and Stegun (1965) for the confluent hypergeometric function $U(a,b,z)$. Hence, the density of V for $v \in \mathbf{R}$ is

$$g_1(v) = \frac{e^{-|v|}|v|^{2p-1}}{\Gamma(p)}U(p,2p,2|v|), v \in \mathbf{R}. \tag{2.1}$$

For $0 = t_0 < t_1 < ... < t_n$, consider n (≥ 2) increments $X_j = X(t_j) - X(t_{j-1}), j = 1, ..., n$ of the Laplace process which are independent symmetrized gamma(p_j) variates, where $p_j = t_j - t_{j-1}$. The joint density of $X_1, ..., X_n$ is then

$$h(x_1, ..., x_n) = \prod_{j=1}^n g_1(x_j)$$

where $g_1(.)$ is given by (2.1). To get the joint density of $Y_j = X(t_j), j = 1, ..., n$ make the partial sum transformation $Y_k = X_1 + ... + X_k, k = 1, ..., n$. The Jacobian of this transformation is unity and hence the joint density of $(Y_1, ..., Y_n),\ y_j \in \mathbf{R}$ is,

$$\ell(y_1, ..., y_n) = \prod_{k=1}^n \frac{e^{|y_k - y_{k-1}|}\ |y_k - y_{k-1}|^{2p_k-1}}{\Gamma(p_k)}U(p_k, 2p_k, 2|y_k - y_{k-1}|)$$

which is the *FDD* of Laplace process.

Remark 2.1. *In connection with the generalized Laplace distribution, the density (2.1) is well-known and it is stated in terms of the modified Bessel function of the 3^{rd} kind (see. Kotz et al. 2001, for further details).*

We know that the paths of Weiner process are continuous and nowhere differentiable with probability 1. For Laplace process we have the following results.

Theorem 2.2. *The paths of Laplace process $\{X(t), t \geq 0\}$ on the probability space $(\Omega, \mathcal{F}, P)$ are continuous in probability and nowhere differentiable.*

Proof. The path of $X(t)$ is continuous in probability *iff* for every $\delta > 0$ and $t \geq 0$,

$$\lim_{\Delta t \to 0} P\{|X(t+\Delta t) - X(t)| \geq \delta\} = 0.$$

Since $X(t+\Delta t) - X(t)$ has the same distribution as $X(\Delta t)$, by Markov's inequality,

$$\begin{aligned} P\{|X(t+\Delta t) - X(t)| \geq \delta\} &= P\{|X(\Delta t)| \geq \delta\} \\ &\leq \frac{E(|X(\Delta t)|^2)}{\delta^2} \\ &= \frac{V(X(\Delta t))}{\delta^2} \\ &= \frac{2\Delta t}{\delta^2}. \end{aligned}$$

Now, letting $\Delta t \to 0$, we have $P\{|X(t+\Delta t) - X(t)| \geq \delta\} \to 0$, proving the first assertion.

Setting $\Delta X(t) = X(t+\Delta t) - X(t)$, $\Delta X(t) = L\sqrt{2\Delta t}$, where $L \sim \mathcal{L}(0,1)$ and

$$\lim_{\Delta t \to 0} \frac{\Delta X(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{L\sqrt{2\Delta t}}{\Delta t} = \lim_{\Delta t \to 0} \frac{L\sqrt{2}}{\sqrt{\Delta t}} = \pm\infty,$$

depending on the sign of L. Hence, the path of $X(t)$ is nowhere differentiable. □

Acknowledgements: Author thanks the referee for the following suggestions. (i) to review the theoretical and applied advances over the years, that made this paper more meaningful (ii) remark 2.1 and (iii) the simplified proof of continuity of paths. He is grateful to Prof. T S K Moothathu for the suggestion to use confluent hypergeometric function to express the *FDD* of the process. This work was mostly done when the author was in the department of Statistics, University of Kerala with a UGC fellowship. He is grateful to the UGC for the support.

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