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Extended Gamma, Beta and Hypergeometric Functions: Properties and Applications

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ABSTRACT

The main objective of this paper is to study the newly introduced extended gamma function and present some properties of the existing extended beta and hypergeometric functions with their properties such as integral formulas, functional relations, differential formulas, summation formulas, recurrence relations and the Mellin transform. In addition, applications of the extended beta function to statistical distributions have been discussed by providing mean, variance, coefficient of variance, moment generating function, cumulative distribution and reliability function of the new extended beta distribution.

Key words and Phrases: *Gamma function, beta function, hypergeometric functions, pochhammer symbol, beta distribution.*

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1 Introduction

Chaudhry and Zubair, (1994) extended the well-known classical gamma function by setting exponential regularization factor $exp\left(-pt^{-1}\right)$ as follows:

$$\Gamma_p(x) = \int_0^\infty t^{x-1} exp\left(-t - pt^{-1}\right) dt, \tag{1.1}$$

where $Re(p) > 0, Re(x) > 0$.

Chaudhry et al., (1997) presented the following extended Euler's beta function with the exponential regularization factor $exp(-pt^{-1}(1-t)^{-1})$:

$$B_p(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} exp\left(-\frac{p}{t(1-t)}\right) dt, \tag{1.2}$$

where $Re(p) > 0$, $Re(x) > 0$, $Re(y) > 0$.

They (Chaudhry et al., 1997) also show the application of the extended Euler's beta function in equation (1.2) to statistics by establishing the following beta distribution and the mean, variance, moment generating function and cumulative distribution:

$$f(t) = \begin{cases} \frac{1}{B_p(\alpha,\beta)} t^{\alpha-1}(1-t)^{\beta-1} exp\left(-\frac{p}{t(1-t)}\right), & 0 < t < 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $p \geq 0$, $-\infty < \alpha, \beta < \infty$.

Chaudhry et al., (2002) used the extended beta function in equation (1.2) to derive the following extended hypergeometric functions:

$$F_p(\alpha, \beta; \gamma; z) = \sum_{\varpi=0}^{\infty} \frac{B_p(\beta + \varpi, \gamma - \beta)}{B(\beta, \gamma - \beta)} (\alpha)_\varpi \frac{z^\varpi}{\varpi!},$$

where $p \geq 0$, $|z| < 1$, $Re(\alpha) > 0$, $Re(\gamma) > Re(\beta) > 0$, and

$$\Phi_p(\beta; \gamma; z) = \sum_{\varpi=0}^{\infty} \frac{B_p(\beta + \varpi, \gamma - \beta)}{B(\beta, \gamma - \beta)} \frac{z^\varpi}{\varpi!},$$

where $p \geq 0$, $Re(\gamma) > Re(\beta) > 0$.

Applying the methodology of (Saif et al., 2020), the newly modified special functions were given by Kulip et al., (2020) in the following equations:

$$\Gamma_p(x; \xi) = \int_0^\infty t^{x-1} \xi^{\left(-t-pt^{-1}\right)} dt, \tag{1.3}$$

where $Re(p) > 0,\ Re(x) > 0,\ \xi \in \mathbb{R}^+ \backslash \{1\}$,

$$B_p(x, y; \xi) = \int_0^1 t^{x-1}(1-t)^{y-1} \xi^{\left(-\frac{p}{t(1-t)}\right)} dt,$$

where $Re(p) > 0,\ Re(x) > 0,\ Re(y) > 0,\ \xi \in \mathbb{R}^+ \backslash \{1\}$,

$$F_p(\alpha, \beta; \gamma; z; \xi) = \sum_{\varpi=0}^{\infty} \frac{B_p(\beta + \varpi, \gamma - \beta; \xi)}{B(\beta, \gamma - \beta)} (\alpha)_\varpi \frac{z^\varpi}{\varpi!},$$

where $p \geq 0,\ |z| < 1,\ Re(\alpha) > 0,\ Re(\gamma) > Re(\beta) > 0,\ \xi \in \mathbb{R}^+ \backslash \{1\}$, and

$$\Phi_p(\beta; \gamma; z; \xi) = \sum_{\varpi=0}^{\infty} \frac{B_p(\beta + \varpi, \gamma - \beta, \xi)}{B(\beta, \gamma - \beta)} \frac{z^\varpi}{\varpi!},$$

where $p \geq 0,\ Re(\gamma) > Re(\beta) > 0,\ \xi \in \mathbb{R}^+ \backslash \{1\}$.

Barahmah, (2021) studies the following extended special function:

$$B_{p,q}(x, y; \xi) = \int_0^1 t^{x-1}(1-t)^{y-1} \xi^{\left(-\frac{p}{t} - \frac{q}{(1-t)}\right)} dt,$$

where $Re(p) > 0,\ Re(q) > 0,\ Re(x) > 0,\ Re(y) > 0,\ \xi \in \mathbb{R}^+ \backslash \{1\}$,

$$F_{p,q}(\alpha, \beta; \gamma; z; \xi) = \sum_{\varpi=0}^{\infty} \frac{B_{p,q}(\beta + \varpi, \gamma - \beta; \xi)}{B(\beta, \gamma - \beta)} (\alpha)_\varpi \frac{z^\varpi}{\varpi!},$$

where $p, q \geq 0,\ |z| < 1,\ Re(\alpha) > 0,\ Re(\gamma) > Re(\beta) > 0,\ \xi \in \mathbb{R}^+ \backslash \{1\}$, and

$$\Phi_{p,q}(\beta; \gamma; z; \xi) = \sum_{\varpi=0}^{\infty} \frac{B_{p,q}(\beta + \varpi, \gamma - \beta, \xi)}{B(\beta, \gamma - \beta)} \frac{z^\varpi}{\varpi!},$$

where $p, q \geq 0,\ Re(\gamma) > Re(\beta) > 0,\ \xi \in \mathbb{R}^+ \backslash \{1\}$.

The main objective of this paper is to introduce a new extended gamma function, $\Gamma_p^\theta(x; \xi)$ and also to study other properties and applications of the existing extended beta function, $B_{p,q}^{\theta,\psi}(x, y; \xi)$.

Definition 1.1: Using the methodology of (Chaudhry and Zubair, 1994, and Parmar, 2013), the following extended gamma function is introduced:

$$\Gamma_p^\theta(x; \xi) = \int_0^\infty t^{x-1} \xi^{\left(-t - \frac{p}{t^\theta}\right)} dt, \tag{1.4}$$

where $Re(p) > 0$, $Re(x) > 0$, $Re(\theta) > 0$, $\xi \in \mathbb{R}^{+}\backslash\{1\}$.

Definition 1.2: The following extended beta function is defined in (Abubakar, 2022 a)as

$$B_{p,q}^{\theta,\psi}(x,y;\xi) = \int_0^1 t^{x-1}(1-t)^{y-1}\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)}dt, \tag{1.5}$$

where $Re(p) > 0$, $Re(q) > 0$, $Re(x) > 0$, $Re(y) > 0$, $Re(\theta) > 0$, $Re(\psi) > 0$, $\xi \in \mathbb{R}^{+}\backslash\{1\}$.

2 Properties of the extended beta function

Theorem 2.1: (First summation formula)

$$B_{p,q}^{\theta,\psi}(x,1-y;\xi) = \sum_{\varpi=0}^{\infty}\frac{(y)_\varpi}{\varpi!}B_{p,q}^{\theta,\psi}(x+\varpi,1;\xi). \tag{2.1}$$

Proof: The left hand of equation (2.1) can be written as

$$B_{p,q}^{\theta,\psi}(x,1-y;\xi) = \int_0^\infty t^{x-1}(1-t)^{-y}\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)}dt.$$

Applying the extended binomial theorem $(1-t)^{-y} = \sum_{\varpi=0}^{\infty}\frac{(y)_\varpi}{\varpi!}t^\varpi$ and then changing the order of summation and integration, gives

$$\begin{aligned} B_{p,q}^{\theta,\psi}(x,1-y;\xi) &= \sum_{\varpi=0}^{\infty}\frac{(y)_\varpi}{\varpi!}\int_0^1 t^{x-1}(1-t)^{1-1}\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)}dt \\ &= \sum_{\varpi=0}^{\infty}\frac{(y)_\varpi}{\varpi!}B_{p,q}^{\theta,\psi}(x+\varpi,1;\xi). \end{aligned}$$

Theorem 2.2: (Second summation formula)

$$B_{p,q}^{\theta,\psi}(x,y;\xi) = \sum_{\varpi=0}^{\infty}B_{p,q}^{\theta,\psi}(x+\varpi,y+1;\xi). \tag{2.2}$$

Proof: The left hand of equation (2.2) can be written as

$$B_{p,q}^{\theta,\psi}(x,y;\xi) = \int_0^1 t^{x-1}(1-t)^{y}(1-t)^{-1}\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)}dt.$$

Applying the extended binomial theorem $(1-t)^{-1}=\sum_{\varpi=0}^{\infty} t^{\varpi}$ and then changing the order of summation and integration, gives

$$\begin{aligned} B_{p,q}^{\theta,\psi}(x,y;\xi) &= \sum_{\varpi=0}^{\infty}\int_0^1 t^{x+\varpi-1}(1-t)^{y+1-1}\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)}dt \\ &= \sum_{\varpi=0}^{\infty} B_{p,q}^{\theta,\psi}(x+\varpi,y+1;\xi). \end{aligned}$$

Theorem 2.3: (Functional relation)

$$B_{p,q}^{\theta,\psi}(x,y;\xi)=B_{p,q}^{\theta,\psi}(x+1,y;\xi)+B_{p,q}^{\theta,\psi}(x,y+1;\xi). \tag{2.3}$$

Proof: The left hand side of equation (2.3) can be expressed as

$$\begin{aligned} B_{p,q}^{\theta,\psi}(x,y;\xi) &= \int_0^1 t^x(1-t)^y\left\{t^{-1}(1-t)^{-1}\right\}\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)}dt \\ &= \int_0^1 t^x(1-t)^y\left\{(1-t)^{-1}+t^{-1}\right\}\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)}dt \\ &= B_{p,q}^{\theta,\psi}(x+1,y;\xi)+B_{p,q}^{\theta,\psi}(x,y+1;\xi). \end{aligned}$$

The following summation formula can obtained by using equation (2.3) and mathematical induction:

Corollary 2.1: (Third summation formula)

$$B_{p,q}^{\theta,\psi}(x,y;\xi)=\sum_{n=0}^{\varpi}\binom{\varpi}{n}B_{p,q}^{\theta,\psi}(x+n,y+\varpi-n;\xi), \qquad \varpi\in\mathbb{N}. \tag{2.4}$$

Theorem 2.4: (Integral representations)

$$\begin{aligned} B_{p,q}^{\theta,\psi}(x,y;\xi) &= 2\int_0^{\frac{\pi}{2}}\cos^{2x-1}\varrho\sin^{2y-1}\varrho\ \xi^{\left(-p\sec^{2\theta}\varrho-q\csc^{2\psi}\varrho\right)}d\varrho, && (2.5)\\ &= \int_0^{\infty}\mu^{x-1}(1+\mu)^{-(x+y)}\xi^{\left(-p(1+\frac{1}{\mu})^{\theta}-q(1+\mu)^{\psi}\right)}d\mu, && (2.6)\\ &= (b-a)^{1-(x+y)}\int_a^b(\mu-a)^{x-1}(b-\mu)^{y-1}\xi^{\left(-\frac{(b-a)^\theta p}{(1+\mu)^\theta}-\frac{(b-a)^\psi q}{(1-\mu)^\psi}\right)}d\mu, && (2.7)\\ &= 2^{1-(x+y)}\int_{-1}^1(\mu+1)^{x-1}(1-\mu)^{y-1}\xi^{\left(-\frac{2^{\eta\theta}p}{(1+\mu)^\theta}-\frac{2^\psi q}{(1-\mu)^\psi}\right)}d\mu, && (2.8)\\ &= 2\int_0^1\mu^{2x-1}(1-\mu^2)^{y-1}\xi^{\left(-\frac{p}{\mu^{2\theta}}-\frac{q}{(1-\mu^2)^\psi}\right)}d\mu. && (2.9) \end{aligned}$$

Proof: Equations (2.5)-(2.9) can be obtained by using the transformation $t = \cos^2 \varrho$, $t = \mu(1+\mu)^{-1}$, $t = (\mu - a)(b-a)^{-1}$, $t = 2^{-1}(1+\mu)$ and $t = \mu^2$ in (1.5), respectively.

Theorem 2.5: The following gamma product formula is correct

$$\Gamma_p^{\theta}(x;\xi)\Gamma_q^{\psi}(y;\xi) = 2\int_0^{\infty} r^{2(x+y)-1}\xi^{\left(-r^2\right)} B^{\theta,\psi}_{\frac{p}{r^2},\frac{q}{r^2}}(x,y;\xi)dr. \tag{2.10}$$

Proof: Putting $t = \kappa^2$ and $t = \lambda^2$ into equation (1.4), the following can obtain:

$$\Gamma_p^{\theta}(x;\xi) = 2\int_0^{\infty} \kappa^{2x-1}\xi^{\left(-\kappa^2-\frac{p}{\kappa^{2\theta}}\right)}d\kappa, \tag{2.11}$$

and

$$\Gamma_q^{\psi}(y;\xi) = 2\int_0^{\infty} \lambda^{2y-1}\xi^{\left(-\lambda^2-\frac{q}{\lambda^{2\psi}}\right)}d\lambda. \tag{2.12}$$

By multiplying equations (2.11) and (2.12) and then setting $\kappa = r\cos\varrho$ and $\lambda = r\sin\varrho$, yields

$$\begin{aligned}&\Gamma_p^{\theta}(x;\xi)\Gamma_q^{\psi}(y;\xi)\\ &= 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} r^{2(x+y)-1}\xi^{\left(-r^2\right)}\cos^{2x-1}\varrho\sin^{2y-1}\xi^{\left(-\frac{p}{r^2\cos^{2\theta}\varrho}-\frac{q}{r^2\sin^{2\psi}\varrho}\right)}drd\varrho.\end{aligned}$$

Reversing the order of integrations leads to

$$\begin{aligned}&\Gamma_p^{\theta}(x;\xi)\Gamma_q^{\psi}(y;\xi)\\ &= 2\int_0^{\infty} r^{2(x+y)-1}\xi^{\left(-r^2\right)}\left\{2\int_0^{\frac{\pi}{2}}\cos^{2x-1}\varrho\sin^{2y-1}\varrho\xi^{\left(-\frac{p}{r^2\cos^{2\theta}\varrho}-\frac{q}{r^2\sin^{2\psi}\varrho}\right)}d\rho\right\}dr\\ &= 2\int_0^{\infty} r^{2(x+y)-1}\xi^{\left(-r^2\right)}B^{\theta,\psi}_{\frac{p}{r^2},\frac{q}{r^2}}(x,y;\xi)dr.\end{aligned}$$

Theorem 2.6: (The Mellin transform)

$$\mathbb{M}\{B^{\theta,\psi}_{p,q}(x,y;\xi);p\to r_1,q\longrightarrow r_2\} = \Gamma(r_1;\xi)\Gamma(r_2;\xi)B(x+\theta r_1,y+\psi r_2), \tag{2.13}$$

where $Re(x+\theta r_1) > 0$, $Re(y+\psi r_2) > 0$, $Re(r_1) > 0$, $Re(r_2) > 0$, $\xi \in \mathbb{R}^+\backslash\{1\}$.

Proof: Using the definition of the Mellin transform (Debnath and Bhatta, 2007) and equation (1.5), yields

$$\begin{aligned}&\mathbb{M}\{B^{\theta,\psi}_{p,q}(x,y;\xi);p\to r_1,q\longrightarrow r_2\}\\ &= \int_0^{\infty}\int_0^{\infty} p^{r_1-1}q^{r_2-1}\left\{\int_0^1 t^{x-1}(1-t)^{y-1}\xi^{\left(-\frac{p}{t^{\theta}}-\frac{q}{(1-t)^{\psi}}\right)}dt\right\}dpdq.\end{aligned}$$

Changing the order of integration gives

$$\mathbb{M}\{B_{p,q}^{\theta,\psi}(x,y;\xi);p\to r_1,q\longrightarrow r_2\}$$
$$=\int_0^1 t^{x-1}(1-t)^{y-1}\left\{\int_0^\infty p^{r_1-1}\xi^{(-\frac{p}{t^\theta})}dp\right\}\left\{\int_0^\infty q^{r_2-1}\xi^{(-\frac{q}{(1-t)^\psi})}dq\right\}dt.$$

Substituting $p=s_1t^\theta$ and $q=s_2(1-t)^\psi$, leads to

$$\mathbb{M}\{B_{p,q}^{\theta,\psi}(x,y;\xi);p\to r_1,q\longrightarrow r_2\}$$
$$=\int_0^1 t^{x+\theta r_1-1}(1-t)^{y+\psi r_2-1}\left\{\int_0^\infty s_1^{r_1-1}\xi^{(-s_1)}ds_1\right\}\left\{\int_0^\infty s_2^{r_2-1}\xi^{(-s_2)}ds_2\right\}dt$$
$$=\Gamma(r_1;\xi)\Gamma(r_2;\xi)B(x+\theta r_1,y+\psi r_2).$$

Considering equation (2.13), the following result can be obtained:

Corollary 2.2: (The inverse Melling transform)

$$B_{p,q}^{\theta,\psi}(x,y;\xi)$$
$$=-\frac{1}{4\pi^2}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\int_{\gamma_2-i\infty}^{\gamma_2+i\infty}\frac{\Gamma(r_1;\xi)\Gamma(r_2;\xi)\Gamma(x+\theta r_1)\Gamma(y+\psi r_2)}{\Gamma(x+y+\theta r_1+\psi r_2)}p^{-r_1}q^{-r_2}dr_1dr_2.$$

where $\gamma_1,\ \gamma_1>0$.

Theorem 2.7: (Recurrence relation)

$$xB_{p,q}^{\theta,\psi}(x,y+1;\xi)-yB_{p,q}^{\theta,\psi}(x+1,y;\xi)$$
$$=In(\xi)\left\{qB_{p,q}^{\theta,\psi}(x+1,y-1;\xi)-pB_{p,q}^{\theta,\psi}(x-1,y+1;\xi)\right\},\tag{2.14}$$

where $Re(x)>0,Re(y)>0,Re(p)>0,Re(q)>0,\xi\in\mathbb{R}^+$.

Proof : Setting

$$B_{p,q}^{\theta,\psi}(x,y;\xi)=\mathbb{M}\{h_{p,q}^{\theta,\psi}(t:y;\xi)\},$$

where

$$h_{p,q}^{\theta,\psi}(t:y;\xi)=H(1-t)(1-t)^{y-1}\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)},\tag{2.15}$$

and $H(1-t)$ is the Heaviside delta defined in (Ata and Kiymaz, 2020) by

$$H(1-t)=\begin{cases}0, & t>0,\\ 1, & t<0.\end{cases}\tag{2.16}$$

Differentiating equation (2.15) partially with respect to t, gives

$$\frac{\partial}{\partial t}h_{p,q}^{\theta,\psi} = -\delta(1-t)(1-t)(1-t)^{y-1}\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)} - H(1-t)(1-t)^{y-2}\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)}$$
$$+ H(1-t)(1-t)^{y-1}In(\xi)\left(\frac{p\theta}{t^{\theta+1}} - \frac{q\psi}{(1-t)^{\psi+1}}\right)\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)}.$$

Using the fact that (Abubakar, 2021 a)

$$\frac{d}{dt}H(1-t) = -\delta(1-t), \tag{2.17}$$

and

$$\delta(1-t) = \delta(t-1) = 0, \quad \text{for } t \neq 0 \tag{2.18}$$

yields

$$\frac{\partial}{\partial t}h_{p,q}^{\theta,\psi} = -H(1-t)(1-t)^{y-2}\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)}$$
$$+ \theta pH(1-t)(1-t)^{y-\theta-2}In(\xi)\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)}$$
$$- \psi qH(1-t)(1-t)^{y-\psi-2}In(\xi)\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)}.$$

Consider the Mellin transform property (Abubakar, 2021 b)

$$\mathbb{M}\{h(t);x\} = F(x), \tag{2.19}$$

and

$$\mathbb{M}\{h(t);x\} = -(x-1)F(x-1), \tag{2.20}$$

leads to

$$-(x-1)B_{p,q}^{\theta,\psi}(x-1,y;\xi) = -(y-1)B_{p,q}^{\theta,\psi}(x,y-1;\xi)$$
$$+ \theta pIn(\xi)B_{p,q}^{\theta,\psi}(x-\theta-1,y;\xi) - \psi qIn(\xi)B_{p,q}^{\theta,\psi}(x,y-\psi-1;\xi).$$

Simplifying, for

$$(x-1)B_{p,q}^{\theta,\psi}(x-1,y;\xi) + (y-1)B_{p,q}^{\theta,\psi}(x,y-1;\xi)$$
$$= In(\xi)\{\psi qB_{p,q}^{\theta,\psi}(x,y-\psi-1;\xi) - \theta pB_{p,q}^{\theta,\psi}(x-\theta-1,y;\xi)\}.$$

Setting $x \to x+1$ and $y \to y+1$ yields the desired result in equation (2.14).
Theorem 2.8: (Differential formula)

$$\frac{\partial^{\varpi}}{\partial x^{\varpi}} B_{p,q}^{\theta,\psi}(x,y;\xi) = \int_0^1 t^{x-1}(1-t)^{y-1} In^{\varpi}(t) \xi^{\left(-\frac{p}{t^{\theta}}-\frac{q}{(1-t)^{\psi}}\right)} dt \tag{2.21}$$

Proof: Differentiating equation (1.5) partially with respect to x, gives

$$\frac{\partial^{\varpi}}{\partial x^{\varpi}} B_{p,q}^{\theta,\psi}(x,y;\xi) = \int_0^1 \left\{\frac{\partial^{\varpi}}{\partial^{\varpi}} t^x\right\} t^{-1}(1-t)^{y-1} In^{\varpi}(t) \xi^{\left(-\frac{p}{t^{\theta}}-\frac{q}{(1-t)^{\psi}}\right)} dt.$$

Using the formula in (Abubakar and Kabara, 2021)

$$\left\{\frac{\partial^{\varpi}}{\partial x^{\varpi}} t^x\right\} = t^x In^{\varpi}(t),$$

yields the desired result in equation (2.21), following the same procedure leads to the corollaries below:

Corollary 2.3:

$$\frac{\partial^{\omega}}{\partial y^{\omega}} B_{p,q}^{\theta,\psi}(x,y;\xi) = \int_0^1 t^{x-1}(1-t)^{y-1} In^{\omega}(1-t) \xi^{\left(-\frac{p}{t^{\theta}}-\frac{q}{(1-t)^{\psi}}\right)} dt,$$

and

$$\frac{\partial^{\varpi+\omega}}{\partial x^{\varpi} y^{\omega}} B_{p,q}^{\theta,\psi}(x,y;\xi) = \int_0^1 t^{x-1} In^{\varpi}(t)(1-t)^{y-1} In^{\omega}(1-t) \xi^{\left(-\frac{p}{t^{\theta}}-\frac{q}{(1-t)^{\psi}}\right)} dt.$$

Corollary 2.4:

$$\frac{\partial^{\varpi}}{\partial p^{\varpi}} B_{p,q}^{\theta,\psi}(x,y;\xi) = (-1)^{\varpi} In^{\varpi}(\xi) B_{p,q}^{\theta,\psi}(x-\theta\varpi,y;\xi),$$

and

$$\frac{\partial^{\omega}}{\partial q^{\omega}} B_{p,q}^{\theta,\psi}(x,y;\xi) = (-1)^{\omega} In^{\omega}(\xi) B_{p,q}^{\theta,\psi}(x,y-\psi\omega;\xi).$$

3 The extended beta distribution

The extended beta distribution is defined as follows:

$$f(t) = \begin{cases} \frac{1}{B_{p,q}^{\theta,\psi}(\alpha,\beta;\xi)} t^{\alpha-1}(1-t)^{\beta-1} \xi^{\left(-\frac{p}{t^{\theta}}-\frac{q}{(1-t)^{\psi}}\right)}, & 0<t<1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

where $p, q \geq 0$, $-\infty < \alpha, \beta < \infty$.

The moment of the distribution in equation (3.1) is

$$E(X^{\varpi}) = \frac{B_{p,q}^{\theta,\psi}(\alpha+\varpi, \beta; \xi)}{B_{p,q}^{\theta,\psi}(\alpha, \beta; \xi)},$$

where $p, q \geq 0$, $-\infty < \alpha, \beta < \infty$, $\varpi \in \mathbb{N}$.

The variance of the distribution in equation (3.1) is given by

$$\begin{aligned} \delta^2 &= E(X^2) - \{E(X)\}^2 \\ &= \frac{B_{p,q}^{\theta,\psi}(\alpha, \beta; \xi) B_{p,q}^{\theta,\psi}(\alpha+2, \beta; \xi) - \{B_{p,q}^{\theta,\psi}(\alpha+1, \beta; \xi)\}^2}{\{B_{p,q}^{\theta,\psi}(\alpha, \beta; \xi)\}^2}. \end{aligned}$$

The Coefficient of Variance (C.V) of the distribution in equation (3.1) is

$$C.V = \sqrt{\frac{B_{p,q}^{\theta,\psi}(\alpha, \beta; \xi) B_{p,q}^{\theta,\psi}(\alpha+2, \beta; \xi)}{B_{p,q}^{\theta,\psi}(\alpha+1, \beta; \xi)} - 1}.$$

The moment generating function (m.g.f) is expressed as

$$M_X(t) = \sum_{\varpi=0}^{\infty} \frac{t^{\varpi}}{\varpi!} E(X^{\varpi}) = \sum_{\varpi=0}^{\infty} \frac{t^{\varpi}}{\varpi!} \frac{B_{p,q}^{\theta,\psi}(\alpha+\varpi, \beta; \xi)}{B_{p,q}^{\theta,\psi}(\alpha, \beta; \xi)}.$$

The characteristic function is given by

$$M_X(e^{it}) = \sum_{\varpi=0}^{\infty} \frac{i^{\varpi} t^{\varpi}}{\varpi!} E(X^{\varpi}) = \sum_{\varpi=0}^{\infty} \frac{i^{\varpi} t^{\varpi}}{\varpi!} \frac{B_{p,q}^{\theta,\psi}(\alpha+\varpi, \beta; \xi)}{B_{p,q}^{\theta,\psi}(\alpha, \beta; \xi)}.$$

The cumulative distribution function, also known as the probability distribution function is

$$F(x) = F(X < x) = \int_0^x f(t)dt = \frac{B_{p,q}^{\theta,\psi;x}(\alpha, \beta; \xi)}{B_{p,q}^{\theta,\psi}(\alpha, \beta; \xi)},$$

where $B_{p,q}^{\theta,\psi;x}(\alpha, \beta; \xi)$ is the lower incomplete beta function defined as

$$B_{p,q}^{\theta,\psi;x}(\alpha, \beta; \xi) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1} \xi^{\left(-\frac{p}{t^{\theta}} - \frac{q}{(1-t)^{\psi}}\right)} dt,$$

where $p, q \geq 0$, $-\infty < \alpha, \beta < \infty$.

The complement of the cumulative distribution function is called the reliability function which is determined by

$$R(x) = F(X > x) = 1 - F(x) = \int_x^1 f(t)dt = \frac{B_{p,q;x}^{\theta,\psi}(\alpha, \beta; \xi)}{B_{p,q}^{\theta,\psi}(\alpha, \beta; \xi)},$$

where $B^{\theta,\psi}_{p,q;x}(\alpha,\beta;\xi)$ is the upper incomplete beta function defined as

$$B^{\theta,\psi}_{p,q;x}(\alpha,\beta;\xi)=\int_x^1 t^{\alpha-1}(1-t)^{\beta-1}\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)}dt,$$

where $p,q\geq 0$, $-\infty<\alpha,\beta<\infty$.

4 Extended hypergeomtric functions

Definition 4.1: The extended Gauss hypergeometric function is given in (Abubakar, 2022 b) as

$$F^{\theta,\psi}_{p,q}(\alpha,\beta;\gamma;z;\xi)=\sum_{\varpi=0}^{\infty}\frac{B^{\theta,\psi}_{p,q}(\beta+\varpi,\gamma-\beta;\xi)}{B(\beta,\gamma-\beta)}(\alpha)_\varpi\frac{z^\varpi}{\varpi!}, \tag{4.1}$$

where $p,q\geq 0, |z|<1, Re(\alpha)>0, Re(\gamma)>Re(\beta)>0$.

Definition 4.2: The extended Kumar confluent function is defined in (Abubakar, 2022 b) by

$$\Phi^{\theta,\psi}_{p,q}(\beta;\gamma;z;\xi)=\sum_{\varpi=0}^{\infty}\frac{B^{\theta,\psi}_{p,q}(\beta+\varpi,\gamma-\beta;\xi)}{B(\beta,\gamma-\beta)}\frac{z^\varpi}{\varpi!}, \tag{4.2}$$

where $p,q\geq 0, Re(\gamma)>Re(\beta)>0$.

Theorem 4.1: (Integral formulas)

$$\begin{aligned}&F^{\theta,\psi}_{p,q}(\alpha,\beta;\gamma;z;\xi)\\&=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\left\{\sum_{\varpi=0}^{\infty}(\alpha)_\varpi\frac{(tz)^\varpi}{\varpi!}\right\}\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)}dt,\end{aligned} \tag{4.3}$$

where $p,q>0; p=q=0$ and $|z|<1, Re(\alpha)>0, Re(\gamma)>Re(\beta)>0$, and

$$\begin{aligned}&F^{\theta,\psi}_{p,q}(\alpha,\beta;\gamma;z;\xi)\\&=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)}dt,\end{aligned} \tag{4.4}$$

where $p,q>0$; $p=q=0$ and $|arg(1-z)|<\pi$, $Re(\alpha)>0$, $Re(\gamma)>Re(\beta)>0$.

Proof: Equation (4.3) can be obtained by using (1.5) in (4.1), using extended binomial theorem $(1-t)^{-y}=\sum_{\varpi=0}^{\infty}\frac{(y)_\varpi}{\varpi!}t^\varpi$ in (4.3), (4.4) is obtained. By putting

$t = \mu(1+\mu)^{-1}$ and $t = \sin^2 \varrho$ into (4.5), the following result can be obtained:

Corollary 4.1:

$$F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi)$$
$$= \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 \mu^{\beta-1}(1+\mu)^{\alpha-\gamma}(1+\mu(1-z))^{-\alpha} \xi^{\left(-p\left(1+\frac{1}{\mu}\right)^{\theta} - q(1+\mu)^{\psi}\right)} d\mu,$$

where $p, q > 0$; $p = q = 0$ and $|arg(1-z)| < \pi$, $Re(\alpha) > 0$, $Re(\gamma) > Re(\beta) > 0$, and

$$F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi)$$
$$= \frac{2}{B(\beta,\gamma-\beta)} \int_0^{\frac{\pi}{2}} \sin^{2\beta-1}\varrho \cos^{2(\gamma-\beta)-1}\varrho (1-z\sin^2\varrho)^{-\alpha} \xi^{\left(-p\sec^{2\theta}\varrho - q\csc^{2\psi}\varrho\right)} d\varrho,$$

where $p, q > 0$; $p = q = 0$ and $|arg(1-z)| < \pi$, $Re(\alpha) > 0$, $Re(\gamma) > Re(\beta) > 0$.

Corollary 4.2:

$$\Phi_{p,q}^{\theta,\psi}(\beta;\gamma;z;\xi)$$
$$= \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} e^{(zt)} \xi^{\left(-\frac{p}{t^{\theta}} - \frac{q}{(1-t)^{\psi}}\right)} dt.$$

Theorem 4.2: (Differential formula)

$$\frac{d^n}{dt^n} F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi) = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} F_{p,q}^{\theta,\psi}(\alpha+n,\beta+n;\gamma+n;z;\xi). \tag{4.5}$$

Proof: From equation (4.1), give

$$\frac{d^n}{dt^n} F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi) = \sum_{\varpi=0}^{\infty} \frac{F_{p,q}^{\theta,\psi}(\beta+\varpi,\gamma-\beta;\xi)}{B(\beta,\gamma-\beta)} \frac{(\alpha)_\varpi}{\varpi!} \frac{d^n}{dt^n} z^{\varpi}.$$

Using the formula in (Srivastava and Manocha, 1984)

$$\frac{d^n}{dt^n} z^{\varpi} = \frac{\Gamma(\varpi+1)}{\Gamma(\varpi-n+1)} z^{\varpi-n}$$

gives

$$\frac{d^n}{dt^n} F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi) = \sum_{\varpi=0}^{\infty} \frac{F_{p,q}^{\theta,\psi}(\beta+\varpi,\gamma-\beta;\xi)}{B(\beta,\gamma-\beta)} \frac{(\alpha)_\varpi}{\Gamma(\varpi-n+1)} z^{\varpi-n}.$$

Setting $\varpi \to \varpi + n$, leads to

$$\frac{d^n}{dt^n} F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi) = \sum_{\varpi=0}^{\infty} \frac{F_{p,q}^{\theta,\psi}(\beta+\varpi+n,\gamma-\beta;\xi)}{B(\beta,\gamma-\beta)} (\alpha)_{\varpi+n} \frac{z^{\varpi}}{\varpi!}.$$

Consider the fact that in (Srivastava and Manocha, 1984)

$$B(\beta,\gamma-\beta)=\frac{(\gamma)_n}{(\beta)_n}B(\beta+n,\gamma-\beta),$$

and

$$(\gamma)_{\varpi+n}=(\alpha)_n(\alpha+n)_{\varpi},$$

yields

$$\begin{aligned}\frac{d^n}{dt^n}F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi)&=\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\sum_{\varpi=0}^{\infty}\frac{F_{p,q}^{\theta,\psi}(\beta+\varpi+n,\gamma-\beta;\xi)}{B(\beta+n,\gamma-\beta)}(\alpha+n)_{\varpi}\frac{z^{\varpi}}{\varpi!}\\&=\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F_{p,q}^{\theta,\psi}(\alpha+n,\beta+n;\gamma+n;z;\xi).\end{aligned}$$

The following result is obtained from the differential formula in equation (4.5):

Corollary 4.3:

$$\frac{d^n}{dt^n}\Phi_{p,q}^{\theta,\psi}(\beta;\gamma;z;\xi)=\frac{(\beta)_n}{(\gamma)_n}\Phi_{p,q}^{\theta,\psi}(\beta+n;\gamma+n;z;\xi).$$

Theorem 4.3: (The Mellin transform)

$$\begin{aligned}&\mathbb{M}\{F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;\xi);p\to r_1,q\longrightarrow r_2\}\\&=\frac{\Gamma(r_1;\xi)\Gamma(r_2;\xi)B(\beta+\theta r_1,\gamma-\beta+\psi r_2)}{B(\beta,\gamma-\beta)}{}_2F_1(\alpha,\beta+\theta r_1;\gamma+\theta r_1+\psi r_2;z),\end{aligned}\tag{4.6}$$

where $Re(\alpha)>0$, $Re(x+\theta r_1)>0$, $Re(y+\psi r_2)>0$, $Re(r_1)>0$, $Re(r_2)>0$, $\xi\in\mathbb{R}^+$.

Proof: Using the definition of the Mellin transform (Debnath and Bhatta, 2007) to equation (4.1) yields

$$\begin{aligned}&\mathbb{M}\{F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;\xi);p\to r_1,q\longrightarrow r_2\}=\int_0^{\infty}\int_0^{\infty}p^{r_1-1}q^{r_2-1}\\&\times\left\{\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\xi^{\left(-\frac{p}{t^{\theta}}-\frac{q}{(1-t)^{\psi}}\right)}dt\right\}dpdq,\end{aligned}$$

Changing the order of integrations gives

$$\begin{aligned}\mathbb{M}\{F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;\xi);p\to r_1,q\longrightarrow r_2\}&=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\\&\times\left\{\int_0^{\infty}p^{r_1-1}\xi^{\left(-\frac{p}{t^{\theta}}\right)}dp\right\}\left\{\int_0^{\infty}q^{r_2-1}\xi^{\left(-\frac{q}{(1-t)^{\psi}}\right)}dq\right\}dt.\end{aligned}$$

Substituting $p = s_1 t^\theta$ and $q = s_2(1-t)^\psi$, leads to

$$\mathbb{M}\{F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;\xi);p \to r_1, q \longrightarrow r_2\} = \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 t^{\beta+\theta r_1-1}(1-t)^{\gamma-\beta+\psi r_2-1} \times (1-tz)^{-\alpha}\left\{\int_0^\infty s_1^{r_1-1}\xi^{(-s_1)}ds_1\right\}\left\{\int_0^\infty s_2^{r_2-1}\xi^{(-s_2)}ds_2\right\}dt$$

Applying the extended gama function in equation (1.4), gives

$$\begin{aligned}&\mathbb{M}\{F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;\xi);p \to r_1, q \longrightarrow r_2\}\\ &= \frac{\Gamma(r_1;\xi)\Gamma(r_2;\xi)}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta+\theta r_1-1}(1-t)^{\gamma-\beta+\psi r_2-1}(1-tz)^{-\alpha}dt\\ &= \frac{\Gamma(r_1;\xi)\Gamma(r_2;\xi)B(\beta+\theta r_1,\gamma-\beta+\psi r_2)}{B(\beta,\gamma-\beta)}{}_2F_1(\alpha,\beta+\theta r_1;\gamma+\theta r_1+\psi r_2;z).\end{aligned}$$

Considering equations (4.2) and (4.6), the following results can be obtained:

Corollary 4.4:

$$\begin{aligned}&\mathbb{M}\{\Phi_{p,q}^{\theta,\psi}(\beta;\gamma;\xi);p \to r_1, q \longrightarrow r_2\}\\ &= \frac{\Gamma(r_1;\xi)\Gamma(r_2;\xi)B(\beta+\theta r_1,\gamma-\beta+\psi r_2)}{B(\beta,\gamma-\beta)}\Phi(\beta+\theta r_1;\gamma+\theta r_1+\psi r_2;z),\end{aligned}$$

where $Re(x+\theta r_1) > 0$, $Re(y+\psi r_2) > 0$, $Re(r_1) > 0$, $Re(r_2) > 0$, $\xi \in \mathbb{R}^+ \setminus \{1\}$.

Corollary 4.5:

$$F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi) = -\frac{1}{4\pi^2}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\int_{\gamma_2-i\infty}^{\gamma_2+i\infty}\frac{\Gamma(r_1;\xi)\Gamma(r_2;\xi)B(\beta+\theta r_1,\gamma-\beta+\psi r_2)}{B(\beta,\gamma-\beta)} \times {}_2F_1(\alpha,\beta+\theta r_1;\gamma+\theta r_1+\psi r_2;z)p^{-r_1}q^{-r_2}dr_1dr_2,$$

and

$$\Phi_{p,q}^{\theta,\psi}(\beta;\gamma;\xi) = -\frac{1}{4\pi^2}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\int_{\gamma_2-i\infty}^{\gamma_2+i\infty}\frac{\Gamma(r_1;\xi)\Gamma(r_2;\xi)B(\beta+\theta r_1,\gamma-\beta+\psi r_2)}{B(\beta,\gamma-\beta)} \times \Phi(\beta+\theta r_1;\gamma+\theta r_1+\psi r_2;z)p^{-r_1}q^{-r_2}dr_1dr_2,$$

where $\gamma_1, \gamma_1 > 0$.

Theorem 4.4: (Difference formula)

$$\begin{aligned}
&(\beta-1)B(\beta-1,\gamma-\beta+1)F_{p,q}^{\theta,\psi}(\alpha,\beta-1;\gamma-1;z;\xi)\\
&=(\gamma-\beta-1)B(\beta,\gamma-\beta-1)F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma-1;z;\xi)\\
&-\alpha z B(\beta,\gamma-\beta)F_{p,q}^{\theta,\psi}(\alpha+1,\beta;\gamma;z;\xi)\\
&-\theta p In(\xi)B(\beta-\theta-1,\gamma-\beta)F_{p,q}^{\theta,\psi}(\alpha,\beta-\theta-1;\gamma-\theta-1;z;\xi)\\
&+\psi q In(\xi)B(\beta,\gamma-\beta-\psi-1)F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma-\beta-\psi-1;z;\xi).
\end{aligned} \tag{4.7}$$

Proof: Setting

$$B_{p,q}^{\theta,\psi}(\beta,\gamma-\beta;\xi)F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi)=\mathbb{M}\{h_{p,q}^{\alpha,\beta,\gamma}(t:y,p,q;\xi)\},$$

where

$$h_{p,q}^{\alpha,\beta,\gamma}(t:y,p,q;\xi)=H(1-t)(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\xi^{\left(-\frac{p}{t^{\theta}}-\frac{q}{(1-t)^{\psi}}\right)},$$

and $H(1-t)$ is the Heaviside delta defined in equation (2.16).Differentiating $h_{p,q}^{\alpha,\beta,\gamma}(t:y,p,q;\xi)$ partially with respect to t, gives

$$\begin{aligned}
&\frac{\partial}{\partial t}h_{p,q}^{\alpha,\beta,\gamma}(t:y,p,q;\xi)\\
&=-\delta(1-t)(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\xi^{\left(-\frac{p}{t^{\theta}}-\frac{q}{(1-t)^{\psi}}\right)}\\
&-(\gamma-\beta-1)H(1-t)(1-t)^{\gamma-\beta-2}(1-tz)^{-\alpha}\xi^{\left(-\frac{p}{t^{\theta}}-\frac{q}{(1-t)^{\psi}}\right)}\\
&+\alpha z H(1-t)(1-t)^{\gamma-\beta-1}(1-tz)^{-(\alpha+1)}\xi^{\left(-\frac{p}{t^{\theta}}-\frac{q}{(1-t)^{\psi}}\right)}\\
&+H(1-t)(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\left(\frac{p\theta}{t^{\theta+1}}-\frac{q\psi}{(1-t)^{\psi+1}}\right)\xi^{\left(-\frac{p}{t^{\theta}}-\frac{q}{(1-t)^{\psi}}\right)}.
\end{aligned}$$

Using equations (2.17) and (2.18), gives

$$\frac{\partial}{\partial t}h_{p,q}^{\alpha,\beta,\gamma}(t:y,p,q;\xi)$$
$$-(\gamma-\beta-1)H(1-t)(1-t)^{\gamma-\beta-2}(1-tz)^{-\alpha}\xi^{\left(-\frac{p}{t^{\theta}}-\frac{q}{(1-t)^{\psi}}\right)}$$
$$+\alpha z H(1-t)(1-t)^{\gamma-\beta-1}(1-tz)^{-(\alpha+1)}\xi^{\left(-\frac{p}{t^{\theta}}-\frac{q}{(1-t)^{\psi}}\right)}$$
$$+p\theta In(\xi)H(1-t)t^{(\theta+1)}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\xi^{\left(-\frac{p}{t^{\theta}}-\frac{q}{(1-t)^{\psi}}\right)}$$
$$-q\psi In(\xi)H(1-t)(1-t)^{\gamma-\beta-2}(1-tz)^{-\alpha}\xi^{\left(-\frac{p}{t^{\theta}}-\frac{q}{(1-t)^{\psi}}\right)}$$

Using the concept of equations (2.19) and (2.20), yields

$$-(\beta-1)B(\beta-1,\gamma-\beta+1)F_{p,q}^{\theta,\psi}(\alpha,\beta-1;\gamma-1;z;\xi)$$
$$=-(\gamma-\beta-1)B(\beta,\gamma-\beta-1)F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma-1;z;\xi)$$
$$+\alpha zB(\beta,\gamma-\beta)F_{p,q}^{\theta,\psi}(\alpha+1,\beta;\gamma;z;\xi)$$
$$+\theta pIn(\xi)B(\beta-\theta-1,\gamma-\beta)F_{p,q}^{\theta,\psi}(\alpha,\beta-\theta-1;\gamma-\theta-1;z;\xi)$$
$$-\psi qIn(\xi)B(\beta,\gamma-\beta-\psi-1)F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma-\beta-\psi-1;z;\xi).$$

Theorem 4.5:

$$\begin{aligned}&(\beta-1)B(\beta-1,\gamma-\beta+1)\Phi_{p,q}^{\theta,\psi}(\beta-1;\gamma-1;z;\xi)\\&=(\gamma-\beta-1)B(\beta,\gamma-\beta-1)\Phi_{p,q}^{\theta,\psi}(\beta;\gamma-1;z;\xi)\\&-zB(\beta,\gamma-\beta)\Phi_{p,q}^{\theta,\psi}(\beta;\gamma;z;\xi)\\&-\theta pIn(\xi)B(\beta-\theta-1,\gamma-\beta)\Phi_{p,q}^{\theta,\psi}(\beta-\theta-1;\gamma-\theta-1;z;\xi)\\&+\psi qIn(\xi)B(\beta,\gamma-\beta-\psi-1)\Phi_{p,q}^{\theta,\psi}(\beta;\gamma-\beta-\psi-1;z;\xi).\end{aligned}\tag{4.8}$$

Proof: Setting

$$B_{p,q}^{\theta,\psi}(\beta,\gamma-\beta;\xi)\Phi p,q^{\theta,\psi}(\beta;\gamma;z;\xi)=\mathbb{M}\{h_{p,q}^{\alpha,\beta,\gamma}(t:y,p,q;\xi)\},$$

where

$$h_{p,q}^{\alpha,\beta,\gamma}(t:y,p,q;\xi)=H(1-t)(1-t)^{\gamma-\beta-1}e^{(tz)}\xi^{\left(-\frac{p}{t^{\theta}}-\frac{q}{(1-t)^{\psi}}\right)},$$

the needful result is obtained.

Theorem 4.6: (Functional relation)

$$\gamma F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi)=\beta F_{p,q}^{\theta,\psi}(\alpha,\beta+1;\gamma;z;\xi)+(\gamma-\beta)F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma+1;z;\xi). \quad (4.9)$$

Proof: : From the functional relation of the extended beta function in (2.3), one can get

$$\begin{aligned}
&F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi)\\
&=\sum_{\varpi=0}^{\infty}(\alpha)_{\varpi}\left\{\frac{B_{p,q}^{\theta,\psi}(\beta+\varpi+1,\gamma-\beta;\xi)+B_{p,q}^{\theta,\psi}(\beta+\varpi,\gamma-\beta+1;\xi)}{B(\beta,\gamma-\beta)}\right\}\frac{z^{\varpi}}{\varpi!}\\
&=\frac{B(\beta+1,\gamma-\beta)}{B(\beta,\gamma-\beta)}\left\{\sum_{\varpi=0}^{\infty}(\alpha)_{\varpi}\frac{B_{p,q}^{\theta,\psi}(\beta+\varpi+1,\gamma-\beta;\xi)}{B(\beta,\gamma-\beta)}\right\}\frac{z^{\varpi}}{\varpi!}\\
&+\frac{B(\beta,\gamma-\beta+1)}{B(\beta,\gamma-\beta)}\left\{\sum_{\varpi=0}^{\infty}(\alpha)_{\varpi}\frac{B_{p,q}^{\theta,\psi}(\beta+\varpi,\gamma-\beta+1;\xi)}{B(\beta,\gamma-\beta+1)}\right\}\frac{z^{\varpi}}{\varpi!}.
\end{aligned}$$

Simplifying, the desired result in equation (4.9) is obtained.

Corollary 4.6:

$$\gamma\Phi_{p,q}^{\theta,\psi}(\beta;\gamma;z;\xi)=\beta\Phi_{p,q}^{\theta,\psi}(\beta+1;\gamma;z;\xi)+(\gamma-\beta)\Phi_{p,q}^{\theta,\psi}(\beta;\gamma+1;z;\xi).$$

Theorem 4.7: (Summation formula)

$$F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi)=(\gamma-\beta)\sum_{n=0}^{\infty}\frac{(\beta)_n}{(\gamma)_{n+1}}F_{p,q}^{\theta,\psi}(\alpha,\beta+n;\gamma+n+1;z;\xi).$$

Proof: From the extended beta function summation formula in equation (2.1), gives

$$\begin{aligned}
&F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi)\\
&=\sum_{\varpi=0}^{\infty}(\alpha)_{\varpi}\frac{B_{p,q}^{\theta,\psi}(\beta+\varpi+n,\gamma-\beta+1;\xi)}{B(\beta,\gamma-\beta+1)}\frac{z^{\varpi}}{\varpi!}\\
&=\sum_{n=0}^{\infty}\frac{B(\beta+n,\gamma-\beta+1)}{B(\beta,\gamma-\beta)}\left\{\sum_{\varpi=0}^{\infty}(\alpha)_{\varpi}\frac{B_{p,q}^{\theta,\psi}(\beta+\varpi+n,\gamma-\beta+1;\xi)}{B(\beta+n,\gamma-\beta+1)}\frac{z^{\varpi}}{\varpi!}\right\}\\
&=(\gamma-\beta)\sum_{n=0}^{\infty}\frac{(\beta)_n}{(\gamma)_{n+1}}F_{p,q}^{\theta,\psi}(\alpha,\beta+n;\gamma+n+1;z;\xi).
\end{aligned}$$

Corollary 4.7:

$$\Phi_{p,q}^{\theta,\psi}(\beta;\gamma;z;\xi)=(\gamma-\beta)\sum_{n=0}^{\infty}\frac{(\beta)_n}{(\gamma)_{n+1}}\Phi_{p,q}^{\theta,\psi}(\beta+n;\gamma+n+1;z;\xi). \tag{4.10}$$

Using the summation formula of the beta function in equation (2.2), the following results can be obtained:

Corollary 4.8:

$$\begin{aligned}&F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi)\\&=\frac{1}{B(\beta,\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(\beta-\gamma+1)_n}{n!}\frac{(\beta)_n}{(\gamma)_{n+1}}F_{p,q}^{\theta,\psi}(\alpha,\beta+n;\gamma+n+1;z;\xi),\end{aligned}$$

and

$$\begin{aligned}&\Phi_{p,q}^{\theta,\psi}(\beta;\gamma;z;\xi)\\&=\frac{1}{B(\beta,\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(\beta-\gamma+1)_n}{n!}\frac{(\beta)_n}{(\gamma)_{n+1}}\Phi_{p,q}^{\theta,\psi}(\beta+n;\gamma+n+1;z;\xi).\end{aligned}$$

Using the summation formula of the beta function in equation (2.4), the following results can be obtained:

Corollary 4.9:

$$F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi)=\sum_{n=0}^{\varpi}\binom{\varpi}{n}\frac{B(\beta+n,\gamma-\beta+\varpi)}{B(\beta,\gamma-\beta)}F_{p,q}^{\theta,\psi}(\alpha,\beta+n;\gamma+n;z;\xi),$$

and

$$\Phi_{p,q}^{\theta,\psi}(\beta;\gamma;z;\xi)=\sum_{n=0}^{\varpi}\binom{\varpi}{n}\frac{B(\beta+n,\gamma-\beta+\varpi)}{B(\beta,\gamma-\beta)}\Phi_{p,q}^{\theta,\psi}(\beta+n;\gamma+n;z;\xi),$$

where $\varpi\in\mathbb{N}$.

Theorem 4.8:

$$F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi)=(1-z)^{-\alpha}F_{p,q}^{\theta,\psi}\left(\alpha,\gamma-\beta;\gamma;-\frac{z}{1-z};\xi\right), \tag{4.11}$$

where $p,q>0$; $p,q=0$ and $|arg(1-z)|<\pi$, $Re(\gamma)>Re(\beta)>0$.

Proof: Rewrite $t\to 1-t$ in (4.4) and use

$$(1-z[1-t])^{-\alpha}=(1-z)^{-\alpha}\left(1+\frac{tz}{1-z}\right)^{-\alpha},$$

yields

$$F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi) = \frac{(1-z)^{-\alpha}}{B(\beta,\gamma-\beta)}\int_0^1 t^{\gamma-\beta-1}(1-t)^{\beta-1}\left(1+\frac{tz}{1-z}\right)\xi^{\left(-\frac{p}{t^\theta}-\frac{q}{(1-t)^\psi}\right)}dt$$
$$= (1-z)^{-\alpha}F_{p,q}^{\theta,\psi}\left(\alpha,\gamma-\beta;\gamma;-\frac{z}{1-z};\xi\right).$$

Applying similar procedure the following results are obtained:

Corollary 4.10:

$$F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi) = (1-z)^{-\beta}F_{p,q}^{\theta,\psi}\left(\gamma-\alpha,\beta;\gamma;-\frac{z}{1-z};\xi\right),$$

and

$$F_{p,q}^{\theta,\psi}(\alpha,\beta;\gamma;z;\xi) = (1-z)^{\gamma-\beta-\alpha}F_{p,q}^{\theta,\psi}\left(\gamma-\alpha,\gamma-\beta;\gamma;z;\xi\right),$$

where $p,q>0$; $p,q=0$ and $|arg(1-z)|<\pi$, $Re(\gamma)>Re(\beta)>0$.

By putting $z\to 1-z^{-1}$ and $z\to -z(1+z)^{-1}$ in equation (4.11), the following corollary can be obtained:

Corollary 4.11:

$$F_{p,q}^{\theta,\psi}\left(\alpha,\beta;\gamma;1-\frac{1}{z};\xi\right) = z^\alpha F_{p,q}^{\theta,\psi}(\alpha,\gamma-\beta;\gamma;1-z;\xi),$$

where $p,q>0$; $p,q=0$ and $|arg(z)|<\pi$, $Re(\gamma)>Re(\beta)>0$, and

$$F_{p,q}^{\theta,\psi}\left(\alpha,\beta;\gamma;\frac{z}{1+z};\xi\right) = (1+z)^\alpha F_{p,q}^{\theta,\psi}(\gamma-\alpha,\gamma-\beta;\gamma;-z;\xi),$$

where $p,q>0$; $p,q=0$ and $|arg(1+z)|<\pi$, $Re(\gamma)>Re(\beta)>0$.

As a consequence of equation (4.11), the following corollary follows:

Corollart 4.12:

$$\Phi_{p,q}^{\theta,\psi}(\beta;\gamma;z;\xi) = e^z\Phi_{p,q}^{\theta,\psi}\left(\gamma-\beta;\gamma;-z;\xi\right). \tag{4.12}$$

Theorem 4.9:

$$\int_0^\infty z^{\alpha-1}\Phi_{p,q}^{\theta,\psi}(\beta;\gamma;-z;\xi)dz = \Gamma(\alpha)F_{p,q}^{\theta,\psi}\left(\alpha,\gamma-\beta;\gamma;1;\xi\right),$$

where $Re(\gamma)>Re(\beta)>0$, $Re(\alpha)>0$.

Proof: Putting $z\to -z$ into equation (4.12), multiplying the result by $z^{\alpha-1}$ and then integrating with respect to z, bound from 0 to ∞, resulting

$$\int_0^\infty z^{\alpha-1}\Phi_{p,q}^{\theta,\psi}(\beta;\gamma;-z;\xi)dz = \int_0^\infty z^{\alpha-1}e^{-z}\Phi_{p,q}^{\theta,\psi}(\gamma-\beta;\gamma;z;\xi)dz$$
$$= \int_0^\infty z^{\alpha-1}e^{-z}\left\{\sum_{\varpi=0}^{\infty}\frac{B_{p,q}^{\theta,\psi}(\gamma-\beta+\varpi,\beta;\xi)}{B(\beta,\gamma-\beta)\frac{z^\varpi}{\varpi!}}\right\}dz.$$

Reversing the order of integration and summation can lead to

$$\int_0^\infty z^{\alpha-1}\Phi_{p,q}^{\theta,\psi}(\beta;\gamma;-z;\xi)dz = \sum_{\varpi=0}^{\infty}\frac{B_{p,q}^{\theta,\psi}(\gamma-\beta+\varpi,\beta;\xi)}{B(\beta,\gamma-\beta)\varpi!}\int_0^\infty z^{\alpha+\varpi-1}e^{-z}dz$$
$$= \Gamma(\alpha)\sum_{\varpi=0}^{\infty}\frac{B_{p,q}^{\theta,\psi}(\gamma-\beta+\varpi,\beta;\xi)}{B(\beta,\gamma-\beta)}(\alpha)_\varpi\frac{(1)^\varpi}{\varpi!}$$
$$= \Gamma(\alpha)F_{p,q}^{\theta,\psi}\left(\alpha,\gamma-\beta;\gamma;1;\xi\right).$$

Theorem 4.10: (Gauss summation formula)

$$F_{p,q}^{\theta,\psi}\left(\alpha,\beta;\gamma;1;\xi\right) = \sum_{\varpi=0}^{\infty}\frac{B_{p,q}^{\theta,\psi}(\beta+\varpi,\gamma-\beta-\alpha;\xi)}{B(\beta,\gamma-\beta)}(\alpha)_\varpi$$

where $p,q>0$; $p,q=0$ when $Re(\alpha)>0$, $Re(\gamma)>Re(\beta)>0$.

Proof: By substituting $z=1$ into equation (4.4), the desired result is obtained.

5 Conclusion

Extended gamma, beta and hypergeometric functions have been studied and investigated with some of their properties such as integral formulas, functional relation, differential formulas, summation formulas, recurrence relations and the Mellin transform. The applications of the extended beta function to statistical distribution have been discussed by providing the extended beta distribution and the corresponding mean, variance, coefficient of variance, moment generating function, cumulative distribution and reliability function. It is reported in Abubakar (2022 a, b) that the newly introduced special functions reduce to some known gamma, beta, Gauss hypergeometric and confluent hypergeoemtric functions, for example, resulting in Chaudhry and Zubair, (1994), Chaudhry et al., (1997, 2002), Lee et al., (2011), Choi et al., (2014), Sahin, et al., (2018), Kulip et al., (2020) and Barahmah (2021).The special functions under considerations are expected to have more applications in fractional calculus, integral transforms, statistics, astrophysics, physics, engineering, science and technology.

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