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A Class of Tests Based Jointly on Sub-Sample Minimum and Median

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ABSTRACT

In Statistics, nonparametric tests play a very important role when the data sets are not normal. In this paper, we develop a class of nonparametric tests to compare location parameters of two populations. The proposed test statistic is based jointly on minimum and median of the sub-samples. The mean and variance of the test statistic are derived. The test statistic has asymptotic normality under some assumptions. The proposed class of the tests is compared with its existing competitor's w.r.t. Pitman and Bahadur asymptotic relative efficiency. As an illustration, the proposed test is applied to a real life data set. We carried out the Monte Carlo simulation study to find the power and level of significance of the proposed test.

Key words and Phrases: *Nonparametric Tests, Bahadur Efficiency, Shift Parameter, Simulation Study, U-Statistic.*

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1 Introduction

In many experimental settings, it is of interest to test the equality of two populations, for the response variable or the characteristic under study when these populations differ only in their location parameters. For example, in the field of agriculture, to test which fertilizer is more effective in the production of a crop, the usual two sample t-test is used, if the data does not violate the normality condition; otherwise the corresponding non-parametric test for the two sample location problem is applied. More examples can be considered in other fields such as in industrial experiments, pharmaceutical studies etc.

In this paper, we develop a class of non-parametric tests for the two sample location problem. In the literature there are many two sample location tests, namely Wilcoxon-Mann-Whitney test (Wilcoxon, 1945; Mann and Whitney, 1947), Kochar (1978), Deshpande and Kochar (1980, 1982), Stephenson and Ghosh (1985), Ahmad (1996), Kumar (1997, 2015), Xie and Priebe (2000), Öztürk (2001, 2002), Kumar et al. (2003), Kössler and Kumar (2008), Kössler (2010), Shetty and Umarani (2010), Kumar and Chattopadhyay (2013), Kumar and Chawla (2016), Kumar and Goyal (2018), and Kumar and Kumar (2020).

The proposed class of test statistic is based jointly on minimum and median of the sub-samples. This class of the test is given in the Section 2 and its distribution is discussed in Section 3. In Section 4, the proposed class of tests is compared with existing tests in terms of Pitman and Bahadur asymptotic relative efficiency. In Section 5, an illustrative example based on real life data set is given to see the working of the proposed test. Simulation study to compute the estimated power and estimated level of significance of the proposed test is given in Section 6.

2 Proposed Class of Tests

Let X_1, X_2, …, X_{n_1} and Y_1, Y_2, …, Y_{n_2} be the independent random samples of size n_1 and n_2 from two populations with absolutely continuous cumulative distribution functions $F(x)$ and $F(x-\Delta)$, respectively, and Δ is called the shift parameter. If

$\Delta > 0$ then it means the $Y's$ are stochastically greater than the $X's$ and if $\Delta < 0$ then it means $Y's$ are stochastically smaller than the $X's$. Now our motive is to test the null hypothesis:

$$H_0 : \Delta = 0$$

against the alternative hypothesis

$$H_1 : \Delta \neq 0.$$

We consider the following class of U-Statistics for testing H_0 against H_1:

$$T_{2c+1,2d+1} = \left[\binom{n_1}{2c+1}\binom{n_2}{2d+1}\right]^{-1} \sum \Phi(X_{w_1}, X_{w_2}, \ldots, X_{w_{2c+1}}; Y_{z_1}, Y_{z_2}, \ldots, Y_{z_{2d+1}}),$$

where $\Phi(X_1, X_2, \ldots, X_{2c+1}; Y_1, Y_2, \ldots, Y_{2d+1})$ is the kernel defined as:
$\Phi(X_1, X_2, \ldots, X_{2c+1}; Y_1, Y_2, \ldots, Y_{2d+1}) =$

$$= \begin{cases} 1, & \text{if } X_{1:2c+1} \leq Y_{1:2d+1} \text{ and } X_{c+1:2c+1} \leq Y_{d+1:2d+1} \\ 0, & \text{otherwise.} \end{cases}$$

Here c and d are the fixed whole numbers such that $0 \leq c \leq n_1$ and $0 \leq d \leq n_1$. Further $X_{1:2c+1} = \min(X_1, X_2, \ldots, X_{2c+1})$, $Y_{1:2d+1} = \min(Y_1, Y_2, \ldots, Y_{2d+1})$, $X_{c+1:2c+1} = \text{median}(X_1, X_2, \ldots, X_{2c+1})$ and $Y_{d+1:2d+1} = \text{median}(Y_1, Y_2, \ldots, Y_{2d+1})$. The summation in the test statistic $T_{2c+1,2d+1}$ is extended over all possible combinations $(w_1, w_2, \ldots, w_{2c+1})$ of $(2c+1)$ integers chosen from $(1, \ldots, n_1)$ and all possible combinations $(z_1, z_2, \ldots, z_{2d+1})$ of $(2d+1)$ integers chosen from $(1, \ldots, n_2)$.

The statistic $T_{2c+1,2d+1}$ is the U-statistic corresponding to the kernel

$$\Phi(X_1, X_2, \ldots, X_{2c+1}; Y_1, Y_2, \ldots, Y_{2d+1}).$$

The test is to reject the null hypothesis H_0 for large values of the test statistic $T_{2c+1,2d+1}$.

Note that when $c = d = 0$, the test statistics $T_{2c+1,2d+1}$ corresponds Wilcoxon-Mann-Whitney test statistic.

3 The Distribution of Test Statistic $T_{2c+1,2d+1}$

The expectation of the test statistic $T_{2c+1,2d+1}$ is

$$E(T_{2c+1,2d+1}) =$$

$$= \left[\binom{n_1}{2c+1}\binom{n_2}{2d+1}\right]^{-1} \sum E[\Phi(X_{w_1}, X_{w_2}, \ldots, X_{w_{2c+1}}; Y_{z_1}, Y_{z_2}, \ldots, Y_{z_{2d+1}})]$$

$$= P[X_{1:2c+1} \leq Y_{1:2d+1} \text{ and } X_{c+1:2c+1} \leq Y_{d+1:2d+1}]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{v} P[X_{1:2c+1} \leq t \text{ and } X_{c+1:2c+1} \leq v] dP[Y_{1:2d+1} \leq t \text{ and } Y_{d+1:2d+1} \leq v].$$

Under the null hypothesis H_0, the expression of expectation, say, $E_0(T_{2c+1,2d+1})$, is:

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{v} \sum_{s=c+1}^{2c+1}\sum_{r=1}^{s} \frac{(2c+1)!(2d+1)!}{(r!)(s-r)!(2c+1-s)!(d-1)!d!} (F(t))^r (F(v)-F(t))^{s+d-r-1}$$

$$\times (1-F(t))^{2c+d-s-1} dF(t) dF(v)$$

$$E_0(T_{2c+1,2d+1}) = \begin{cases} \frac{1}{2}, & \text{for } c = d = 0 \\ \left[\binom{2c+2d+2}{2d+1}\right]^{-1} \sum_{s=c+1}^{2c+1}\sum_{r=1}^{s} \binom{s-r+d-1}{d-1}\binom{2c+1-s+d}{d}, & \text{for } c, d \geq 1. \end{cases}$$

The following theorem provides asymptotic distribution of test statistic $T_{2c+1,2d+1}$.

Theorem 1: Let $N = n_1 + n_2$. The asymptotic distribution of $N^{\frac{1}{2}}[T_{2c+1,2d+1} - E(T_{2c+1,2d+1})]$, as $N \to \infty$ in such a way that $\frac{n_1}{N} \to \lambda$ and $0 \leq \lambda \leq 1$ is Normal with mean zero and variance $\sigma^2(T_{2c+1,2d+1})$, as

$$\sigma^2(T_{2c+1,2d+1}) = (2c+1)^2 \frac{\xi_{10}}{\lambda} + (2d+1)^2 \frac{\xi_{01}}{1-\lambda},$$

where

$$\xi_{10} = E[(\Phi(x_0, X_2, \ldots, X_{2c+1}; Y_1, Y_2, \ldots, Y_{2d+1}))^2] - [E(T_{2c+1,2d+1})]^2,$$

and

$$\xi_{01} = E[(\Phi(X_1, X_2, \ldots, X_{2c+1}; y_0, Y_2, \ldots, Y_{2d+1}))^2] - [E(T_{2c+1,2d+1})]^2,$$

with

$$\Phi(x_0, X_2, \ldots, X_{2c+1}; Y_1, Y_2, \ldots, Y_{2d+1}) =$$

$$= E[\Phi(X_1, X_2, \ldots, X_{2c+1}; Y_1, Y_2, \ldots, Y_{2d+1})|X_1 = x_0],$$

and

$$\Phi(X_1, X_2, \ldots, X_{2c+1}; y_0, Y_2, \ldots, Y_{2d+1}) =$$

$$= E[\Phi(X_1, X_2, \ldots, X_{2c+1}; y_0, Y_2, \ldots, Y_{2d+1})|Y_1 = y_0].$$

Proof: Follows from the results of Randles and Wolfe (1979), Chapter 3, p. 92.

Under H_0, asymptotic null variance of the test statistic, $\sigma_0^2(T_{2c+1,2d+1})$, after some computation is:

$$\sigma_0^2(T_{2c+1,2d+1}) = \frac{(2c+1)^2(\Psi_{2c+1,2d+1})}{\lambda(1-\lambda)},$$

where

$$\Psi_{2c+1,2d+1} = \int_{-\infty}^{\infty} I^2 dx - (E_0(T_{2c+1,2d+1}))^2,$$

with

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{v} (P[\min(x, X_2, \ldots, X_{2c+1}) \leq t \,\text{and}\, \text{median}(x, X_2, \ldots, X_{2c+1}) \leq v])$$
$$\times (dP[\min(Y_1, Y_2, \ldots, Y_{2d+1}) \leq t \,\text{and}\, \text{median}(Y_1, Y_2, \ldots, Y_{2d+1}) \leq v]).$$

After some computations, we get

$$\Psi_{2c+1,2d+1} = \begin{cases} \frac{1}{12}, & \text{for } c = d = 0 \\ A + B + C + D + E + F - (E_0(T_{2c+1,2d+1}))^2, & \text{for } c, d \geq 1, \end{cases}$$

where

$$A = \left(\binom{2c+2d+1}{2d+1}^{-1} \sum_{s=c+1}^{2c} \sum_{r=1}^{s} \binom{s-r+d-1}{d-1} \binom{2c-s+d}{d} \right)^2,$$

$$B = \left(\frac{(2d+1)!}{d!(d-1)!}\right)^2 \sum_{s=c}^{2c} \sum_{l=0}^{2c+d-s} \sum_{m=0}^{s+d-1} \sum_{p=c}^{2c} \sum_{q=0}^{2c+d-p} \sum_{w=0}^{p+d-1} \frac{(-1)^{m+l+q+w}}{(m+1)(w+1)} \binom{2c}{p}\binom{2c}{s}$$

$$\times \binom{s+d-1}{w}\binom{p+d-1}{w}\binom{2c+d-s-1}{l}\binom{2c+d-p-1}{q}$$

$$\times \left(\left[\frac{1-\frac{1}{s+d+l+2}-\frac{1}{p+d+q+2}+\frac{1}{p+s+2d+l+q+3}}{(s+d+l+1)(p+d+q+1)}\right]\right.$$

$$-\left[\frac{\frac{1}{w+2}-\frac{1}{p+d+q+2}-\frac{1}{s+d+l+w+3}+\frac{1}{p+s+2d+l+q+3}}{(s+d+l+1)(p+d-w+q)}\right]$$

$$-\left[\frac{\frac{1}{m+2}-\frac{1}{s+d+l+2}-\frac{1}{p+d+q+m+3}+\frac{1}{p+s+2d+l+q+3}}{(p+d+q+1)(s+d-m+l)}\right]$$

$$\left.+\left[\frac{\frac{1}{m+w+3}-\frac{1}{p+d+q+m+3}-\frac{1}{s+d+l+w+3}+\frac{1}{p+s+2d+l+q+3}}{(s+d-m+l)(p+d-w+q)}\right]\right),$$

$$C = \left(\frac{(2d+1)!(2c)!}{(c+d)!d!c!}\right)\left(\sum_{r=1}^{c}\binom{c-r+d-1}{d-1}\right)^2 \sum_{m=0}^{c+d}\sum_{n=0}^{c+d}(-1)^{m+n}\binom{c+d}{m}\binom{c+d}{n}$$

$$\times \left[\frac{1-\frac{1}{c+d+m+2}-\frac{1}{c+d+n+2}+\frac{1}{2c+2d+m+n+3}}{(c+d+m+1)(c+d+n+1)}\right],$$

$$D = 2\binom{2c+2d+1}{2d+1}^{-1} \sum_{s=c+1}^{2c}\sum_{r=1}^{s}\binom{s-r+d-1}{d-1}\binom{2c-s+d}{d}\frac{(2d+1)!(2c)!}{(c+d)!d!c!}$$

$$\times \sum_{r=1}^{c}\binom{c-r+d-1}{d-1}\sum_{m=0}^{c+d}(-1)^m\binom{c+d}{m}\frac{1}{c+d+m+2},$$

$$E = 2\left(\frac{(2d+1)!^2(2c)!}{(c+d)!(d)!^2c!(d-1)!}\right)\sum_{r=1}^{c}\binom{c-r+d-1}{d-1}\sum_{n=0}^{c+d}\sum_{s=c}^{2c}\sum_{l=0}^{2c+d-s}\sum_{m=0}^{s+d-1}\frac{(-1)^{n+l+m}}{m+1}$$

$$\times \binom{c+d}{n}\binom{2c}{s}\binom{s+d-1}{m}\binom{2c+d-s}{l}$$

$$\times\left(\left[\frac{1-\frac{1}{s+d+l+2}-\frac{1}{c+d+n+2}+\frac{1}{c+s+2d+l+n+3}}{(s+d+l+1)(c+d+n+1)}\right]\right.$$
$$\left.-\left[\frac{\frac{1}{m+2}-\frac{1}{s+d+l+2}-\frac{1}{c+d+n+m+3}+\frac{1}{c+s+2d+l+n+3}}{(s+d+l-m)(c+d+n+1)}\right]\right),$$

and

$$F=\frac{2}{d!}\binom{2c+2d+1}{2d+1}^{-1}\sum_{s=c+1}^{2c}\sum_{r=1}^{s}\binom{s-r+d-1}{d-1}\binom{2c-s+d}{d}\sum_{s=0}^{2c}\sum_{l=0}^{2c+d-s}\sum_{m=0}^{s+d-1}$$
$$\frac{(2d+1)!}{(d-1)!}\frac{(-1)^{m+l}}{m+1}\binom{2c}{s}\binom{s+d-1}{m}\binom{2c+d-s}{l}\left[\frac{1}{s+d+l+2}-\frac{\frac{1}{m+2}-\frac{1}{s+d+l+2}}{s+d-m+l}\right].$$

In the Table 1, the values of null mean $E_0(T_{2c+1,2d+1})$ and null variance $\sigma_0^2(T_{2c+1,2d+1})$ are provided for some choices of c and d.

Table 1: Expectation and Variance of $T_{2c+1,2d+1}$ under null hypothesis

(c,d)	(1,1)	(2,1)	(3,1)	(1,2)	(2,2)	(3,2)	(1,3)	(2,3)	(3,3)
$E_o(T_{2c+1,2d+1})$	0.3500	0.3928	0.4166	0.2678	0.3214	0.3547	0.2166	0.2714	0.3082
$\sigma_o^2(T_{2c+1,2d+1})$	0.1812	0.2383	0.2752	0.1781	0.2630	0.3251	0.1630	0.2609	0.3403

4 Asymptotic Relative Efficiency

In this section, we compute two types of well-known asymptotic relative efficiency namely Pitman asymptotic relative efficiency (Pitman ARE) and Bahadur asymptotic relative efficiency (Bahadur ARE) of the test statistic $T_{2c+1,2d+1}$. We firstly compute the Pitman ARE of the test statistic.

4.1 Pitman Asymptotic Relative Efficiency

From the definition of Pitman ARE, the limiting efficacy of the test statistic $T_{2c+1,2d+1}$, under the sequence of local alternatives, $\Delta_N=\frac{\Delta}{N^{\frac{1}{2}}}$, is given as:

$$e^2(T_{2c+1,2d+1})=\lim_{N\to\infty}\frac{\frac{d}{d\Delta_N}\left[E(T_{2c+1,2d+1})|\Delta_N=0\right]^2}{N\sigma_0^2(T_{2c+1,2d+1})}.$$

For $c = d = 0$,

$$\frac{d}{d\Delta_N}\left[E(T_{2c+1,2d+1})|\Delta_N = 0\right] = \int_{-\infty}^{\infty} (f(x))^2 dx,$$

and for $c, d \geq 1$,

$$\frac{d}{d\Delta_N}\left[E(T_{2c+1,2d+1})|\Delta_N = 0\right] =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{y} N^{\frac{1}{2}} \left(\frac{(2c+1)!}{c!(c-1)!}\right)\left(\frac{(2d+1)!}{d!(d-1)!}\right)(F(y)-F(x))^{d-1}(1-F(y))^d f(y)f(x)$$

$$\times \sum_{m=0}^{c-1}\sum_{n=0}^{c}\left(\frac{(-1)^{m+n}\binom{c-1}{m}\binom{c}{n}}{c+n-m} - (F(x))^{c+n} f(x)\right.$$

$$\left. + \frac{(c+n-m)(F(y))^{c+n-m-1}(F(x))^{m+1} f(y)}{m+1}\right) dxdy.$$

We now compare the performance of proposed test with respect to some of its competitors such as Wilcoxon-Mann-Whitney test (Wilcoxon (1945) and Mann and Whitney (1947)) test(WMW), Kumar (1997) test(K_m), Öztürk (2001) test ($OZ_{r,s}$) and Kumar (2015) test ($K_{r_1,r_2;i,j}$), in terms of the Pitman asymptotic relative efficiency for various underlying distributions. The values of Pitman asymptotic relative efficiency with respect to Wilcoxon-Mann-Whitney test are given in Table 2. The values of Pitman asymptotic relative efficiency with respect to other tests can be found using efficacy from their respective paper.

Table 2: Pitman asymptotic relative efficiencies of $T_{2c+1,2d+1}$ relative to Wilcoxon-Mann-Whitney (WMW) test

Distribution	*(c,d)*								
	(1,1)	(2,1)	(3,1)	(1,2)	(2,2)	(3,2)	(1,3)	(2,3)	(3,3)
Uniform	1.0344	1.0274	1.0178	1.3746	1.4016	1.4137	1.7177	1.7613	1.7854
U-quadratic	1.1248	1.0486	1.9837	1.9331	1.8549	1.7876	2.8084	2.7056	2.6250
Normal	0.9296	0.9433	0.9425	0.8677	0.9083	0.9238	0.8046	0.8614	0.8882
Cauchy	0.8889	0.9802	1.0517	0.6449	0.7668	0.8578	0.4779	0.6128	0.7137
Exponential	2.0274	1.9424	1.8925	3.3560	3.2736	3.2266	4.7021	4.6126	4.5603
$\beta_1(1,2)$	1.5746	1.5346	1.5082	2.3629	2.3476	2.3370	3.1450	3.1410	3.1377
$\beta_1(2,2)$	0.9592	0.9580	0.9464	1.0110	1.0359	1.0390	1.0438	1.0833	1.0958
Gumbel	1.1917	1.1631	1.1329	1.2707	1.2598	1.2369	1.2884	1.2945	1.2808

$\boldsymbol{\beta}_1(\boldsymbol{p},\, \boldsymbol{q})$ represents the Beta distribution of first kind with parameters $\boldsymbol{p}$ and $\boldsymbol{q}$.

From the Pitman asymptotic relative efficiency comparisons, we have the following observations:

1. The proposed test performs better than WMW test for the Uniform, U-quadratic, $\beta_1(1,2)$, Exponential and Gumbel distributions for all the considered pairs of (c,d).

2. The proposed test performs better from than the class of Kumar (K_m) tests for the Uniform, U-quadratic, $\beta_1(1,2)$, $\beta_1(2,2)$, Exponential, Gumbel distributions for all considered choices of m and (c,d). Additionally, proposed test performs better than class of Kumar (K_m) tests for all the considered choices of m and $(c,d)=(2,1)$. But proposed test not performs better than Kumar (K_m) tests for Cauchy distribution.

3. In comparison with the class of tests of Öztürk $(OZ_{r,s})$, the proposed test performs better for the Exponential and Gumbel distributions for all considered pairs of (r,s) and (c,d). However, for Cauchy distribution, the proposed test performs better than Öztürk $(OZ_{r,s})$ for all the considered pairs (r,s) and with $(c,d)=(3,1)$.

4. The proposed test performs better than the class of tests of Kumar $(K_{r_1,r_2;i,j})$ for the Exponential and Gumbel distributions for all the considered pairs $(r_1,r_2;i,j)$ and (c,d).

4.2 Bahadur Asymptotic Relative Efficiency

The approximate Bahadur slope of the test statistics $T_{2c+1,2d+1}$ is given by

$$C(T_{2c+1,2d+1}) = \frac{1}{\sigma_0^2(T_{2c+1,2d+1})}[E(T_{2c+1,2d+1}) - E_0(T_{2c+1,2d+1})]^2.$$

Bahadur ARE of $T_{2c+1,2d+1}$ with respect to any other test U (say) is given as:

$$B(T_{2c+1,2d+1}) = \frac{C(T_{2c+1,2d+1})}{C(U)},$$

where $C(U)$ is slope of the U test. We now compare performance of the proposed test in terms of Bahadur ARE, relative to all those tests, which we considered for comparison in terms of Pitman ARE for various underlying distributions. The values of Bahadur asymptotic relative efficiency with respect to Wilcoxon-Mann-Whitney test are given in Table 3. The values of Bahadur asymptotic relative efficiency with respect to other tests can be found using efficacy from their respective paper.

Table 3: Bahadur asymptotic relative efficiencies of $T_{2c+1,2d+1}$ relative to Wilcoxon-Mann-Whitney (WMW) test

Distribution	Δ	*(c,d)*								
		(1,1)	(2,1)	(3,1)	(1,2)	(2,2)	(3,2)	(1,3)	(2,3)	(3,3)
Uniform	0.05	1.3812	1.3332	1.2818	1.8371	1.8081	1.7539	2.2964	2.2680	2.2018
	0.10	1.0070	0.9459	0.8863	1.3427	1.2722	1.1899	1.6804	1.5915	1.4817
	0.15	0.9733	0.8920	0.8163	1.3034	1.1861	1.0720	1.6344	1.4783	1.3213
U-quadratic	0.05	0.2357	0.1966	0.1624	0.3807	0.3116	0.2563	0.5356	0.4305	0.3504
	0.10	0.1193	0.1071	0.0715	0.1932	0.1487	0.1100	0.2693	0.1957	0.1448
	0.15	0.0487	0.0834	0.0024	0.0893	0.0818	0.0324	0.1279	0.0962	0.0547
Normal	0.05	0.9479	0.9571	0.9533	0.8978	0.9340	0.9456	0.8415	0.8453	0.9184
	0.10	0.9655	0.9695	0.9623	0.9277	0.9584	0.9653	0.8788	0.9283	0.9467
	0.15	0.9823	0.9806	0.9696	0.9572	0.9812	0.9827	0.9163	0.9602	0.9728
Cauchy	0.05	0.8946	0.9824	1.0516	0.6530	0.7736	0.8589	0.4848	0.6164	0.7148
	0.10	0.9007	0.9834	1.0507	0.6608	0.7736	0.8589	0.4917	0.6194	0.7149
	0.15	0.9062	0.9840	1.0487	0.6684	0.7759	0.8577	0.4985	0.6219	0.7139
Exponential	0.05	0.1243	0.1045	0.0875	0.8340	0.7454	0.6694	1.7276	1.5626	1.4196
	0.10	0.0592	0.0415	0.0277	0.5615	0.4589	0.3762	1.2291	1.0271	0.8628
	0.15	0.0213	0.0104	0.0038	0.3659	0.2712	0.2017	0.8615	0.6644	0.5174
$\beta_1(1,2)$	0.05	1.5703	1.4504	1.3471	2.2996	2.1145	1.9453	3.0308	2.7696	2.5269
	0.10	1.5394	1.3653	1.2308	2.2083	1.8997	1.6588	1.6588	2.4362	2.0897
	0.15	1.4889	1.2746	1.1305	2.0983	1.6992	1.4305	2.7228	2.1363	1.7400
$\beta_1(2,2)$	0.05	1.0393	1.0168	0.9890	1.1619	1.1622	1.1413	1.2567	1.2741	1.2596
	0.10	1.1027	1.0477	0.9985	1.2922	1.2430	1.1833	1.4506	1.4097	1.3430
	0.15	1.1483	1.0515	0.9780	1.3955	1.2738	1.1661	1.6134	1.4769	1.3408
Gumbel	0.05	1.1665	1.1413	1.1136	1.2250	1.2193	1.2010	1.2264	1.2376	1.2291
	0.10	1.1405	1.1182	1.0927	1.1789	1.1774	1.1629	1.1650	1.1798	1.1756
	0.15	1.1138	1.0939	1.0703	1.1326	1.1344	1.1231	1.1046	1.1218	1.1207

$\boldsymbol{\beta}_1(\boldsymbol{p}, \boldsymbol{q})$ represents the Beta distribution of first kind with parameters $\boldsymbol{p}$ and $\boldsymbol{q}$.

From the Bahadur asymptotic relative efficiency comparisons, we note that for the following cases, the performance of $T_{2c+1,2d+1}$ test in terms of Bahadur AREs is same as that of the corresponding Pitman AREs

1. In comparison with the WMW test, when $\Delta = 0.05$ and all the considered pairs (c, d) of the test statistics $T_{2c+1,2d+1}$ for the Normal, Cauchy and Gumbel distributions.

2. In comparison with the class of tests of Kumar (K_m); when $\Delta = 0.05$ with $m = 3$ and $d = 1$, and all the pairs (c,d) of $T_{2c+1,2d+1}$ for the Uniform distribution; when $\Delta = 0.05$ with $m = 1$, and all the pairs (c, d) of $T_{2c+1,2d+1}$ for the Normal distribution; when $\Delta = 0.10$, and all the pairs (c, d) of $T_{2c+1,2d+1}$ for the Cauchy distribution; when $\Delta = 0.05$, and all the pairs (c, d) of $T_{2c+1,2d+1}$ for the Gumbel distribution.

3. In comparison with the class of tests of Öztürk $OZ_{r,s}$ and Kumar $K_{r_1,r_2;i,j}$, when $\Delta = 0.05$ with any choice of pairs(r, s), $(r_1, r_2; i, j)$ and all the pairs (c, d) of $T_{2c+1,2d+1}$ for the Normal, Cauchy and Gumbel distributions.

5 An Illustration

Eriksen et al. (1986) carried out a study for the alcoholics. Two different therapies were given to patients in the alcohol treatment center. In the control group, traditional treatment program was given, while in the treatment group along with traditional treatment program also the classes in social skill training (SST) were given. The alcohol intakes for the one-year in centiliter were recorded for 12 patients in the control group and for 11 patients in the treatment group. To test the null hypothesis H_0 versus the alternative hypothesis H_1 i.e. to test whether SST had effect on the patients, we need to test $H_0 : \Delta = 0$ versus $H_1 : \Delta \neq 0$. Table 4 lists the values of the test statistics $T_{2c+1,2d+1}$ for some pairs of (c, d) and the corresponding p-values.

Table 4: Calculated values of $T_{2c+1,2d+1}$ and the corresponding p-values

(c,d)	(1,1)	(2,1)	(3,1)	(1,2)	(2,2)	(3,2)	(1,3)	(2,3)	(3,3)
$T_{2c+1,2d+1}$	0.9397	0.9886	0.9992	0.9068	0.9810	0.9987	0.8745	0.9734	0.9982
p-value	0.0002	0.0008	0.0020	0.0001	0.0005	0.0017	0.0001	0.0002	0.0015

From the Table 4, it can be noticed that for all the pairs of (c, d), the p-value is less than 0.05 which implies that we reject the null hypothesis. Thus, we conclude that there is significant effect of social skill training on the consumption of alcohol in the patients.

6 Simulation Study

In this Section, we carried out the Monte Carlo simulation study to find the estimated statistical power and estimated level of significance by generating data from Uniform distribution for different sample sizes. The shift parameter considered is $\Delta = 0.1, 0.2, 0.3, 0.4$ and the level of significance is fixed at $\alpha = 0.05, 0.10$. The simulation is repeated 10,000 times based on independent random samples of size 10, 15 and 20. The estimated level of significanceand estimated power of the proposed test are recorded in Tables 5-8.

Table 5: Estimated level of significance for $\alpha = 0.05$

	(c,d)								
n_1, n_2	(1,1)	(2,1)	(3,1)	(1,2)	(2,2)	(3,2)	(1,3)	(2,3)	(3,3)
10,10	0.0478	0.0515	0.0482	0.0624	0.0719	0.0711	0.0640	0.0731	0.0712
15,10	0.0459	0.0456	0.0480	0.0540	0.0567	0.0659	0.0568	0.0591	0.0678
15,15	0.0460	0.0477	0.0491	0.0559	0.0569	0.0620	0.0569	0.0584	0.0650
20,10	0.0480	0.0489	0.0502	0.0541	0.0545	0.0570	0.0531	0.0575	0.0577
20,15	0.0485	0.0492	0.0525	0.0531	0.0538	0.0579	0.0555	0.0562	0.0579
20,20	0.0489	0.0499	0.0520	0.0552	0.0561	0.0567	0.0559	0.0559	0.0559

Table 6: Estimated level of significance for $\alpha = 0.10$

	(c,d)								
n_1, n_2	(1,1)	(2,1)	(3,1)	(1,2)	(2,2)	(3,2)	(1,3)	(2,3)	(3,3)
10,10	0.0995	0.1052	0.1243	0.0968	0.1098	0.1116	0.1073	0.1032	0.1167
15,10	0.1054	0.1075	0.1121	0.0936	0.1087	0.1088	0.0899	0.1099	0.1167
15,15	0.1040	0.1088	0.1101	0.0906	0.1080	0.1076	0.0901	0.1083	0.1109
20,10	0.1025	0.1055	0.1056	0.0994	0.1044	0.1029	0.0991	0.1021	0.1077
20,15	0.1080	0.1024	0.1029	0.0956	0.1028	0.1027	0.0987	0.1001	0.1023
20,20	0.1007	0.1012	0.1015	0.0988	0.1011	0.1010	0.0918	0.1090	0.1019

Table 7: Estimated statistical power of $T_{2c+1,2d+1}$ at $\alpha = 0.05$

		(c,d)								
n_1, n_2	Δ	(1,1)	(2,1)	(3,1)	(1,2)	(2,2)	(3,2)	(1,3)	(2,3)	(3,3)
10,10	0.1	0.1705	0.1681	0.1594	0.2098	0.2246	0.2263	0.2544	0.2596	0.2674
	0.2	0.3787	0.3659	0.3462	0.4596	0.4715	0.4572	0.5204	0.5366	0.5426
	0.3	0.6331	0.6013	0.5715	0.7003	0.7072	0.7145	0.7592	0.7608	0.7649
	0.4	0.8334	0.7991	0.7732	0.8842	0.8853	0.8894	0.9131	0.9211	0.9293
15,10	0.1	0.1706	0.1560	0.1580	0.2312	0.2350	0.2400	0.2587	0.2597	0.2698
	0.2	0.4162	0.4016	0.3732	0.5101	0.5200	0.5260	0.5514	0.5610	0.5690
	0.3	0.6914	0.6773	0.6144	0.7500	0.7535	0.7560	0.8246	0.8297	0.8311
	0.4	0.8789	0.8806	0.8324	0.9385	0.9412	0.9457	0.9542	0.9588	0.9633
15,15	0.1	0.2046	0.1820	0.1765	0.2530	0.2580	0.2596	0.2950	0.2997	0.3014
	0.2	0.4840	0.4655	0.4430	0.5620	0.5635	0.5680	0.6280	0.6330	0.6344
	0.3	0.7810	0.7560	0.7113	0.8330	0.8375	0.8444	0.8610	0.8650	0.8677
	0.4	0.9310	0.9280	0.9130	0.9690	0.9700	0.9740	0.9747	0.9755	0.9788
20,10	0.1	0.1939	0.1930	0.1821	0.2241	0.2339	0.2371	0.2647	0.2697	0.2710
	0.2	0.4369	0.3930	0.3724	0.5296	0.5386	0.5427	0.5917	0.5940	0.5966
	0.3	0.7378	0.6960	0.6700	0.8180	0.8222	0.8299	0.8564	0.8674	0.8698
	0.4	0.9012	0.9000	0.8915	0.9550	0.9456	0.9496	0.9731	0.9733	0.9765
20,15	0.1	0.2084	0.1970	0.1796	0.2565	0.2596	0.2641	0.2920	0.3011	0.3112
	0.2	0.5268	0.4780	0.4185	0.6060	0.6111	0.6187	0.7080	0.7118	0.7148
	0.3	0.8204	0.7690	0.7509	0.8806	0.8910	0.8954	0.9150	0.9201	0.9245
	0.4	0.9517	0.9470	0.9299	0.9793	0.9799	0.9815	0.9801	0.9805	0.9878
20,20	0.1	0.2315	0.2256	0.2178	0.3080	0.3131	0.3165	0.3355	0.3415	0.3455
	0.2	0.5847	0.5420	0.5160	0.6820	0.6824	0.6941	0.7180	0.7225	0.7265
	0.3	0.8664	0.8540	0.7940	0.9200	0.9240	0.9254	0.9312	0.9354	0.9369
	0.4	0.9720	0.9740	0.9570	0.9820	0.9841	0.9910	0.9840	0.9845	0.9898

Table 8: Estimated statistical power of $T_{2c+1,2d+1}$ at α= 0.10

		(c,d)								
n_1, n_2	Δ	(1,1)	(2,1)	(3,1)	(1,2)	(2,2)	(3,2)	(1,3)	(2,3)	(3,3)
10,10	0.1	0.2577	0.2555	0.2459	0.2806	0.3086	0.3293	0.3178	0.3241	0.3333
	0.2	0.4906	0.4885	0.4742	0.5486	0.5566	0.5602	0.6054	0.6312	0.6374
	0.3	0.7210	0.7086	0.7012	0.7784	0.7798	0.7818	0.8166	0.8210	0.8347
	0.4	0.8922	0.8795	0.8629	0.9098	0.9112	0.9178	0.9411	0.9419	0.9456
15,10	0.1	0.2556	0.2463	0.2354	0.3012	0.3108	0.3154	0.3123	0.3200	0.3219
	0.2	0.5272	0.5147	0.5100	0.5901	0.5944	0.6018	0.6298	0.6318	0.6357
	0.3	0.7822	0.7489	0.7421	0.8154	0.8155	0.8198	0.8439	0.8469	0.8479
	0.4	0.9384	0.9245	0.9200	0.9307	0.9333	0.9414	0.9380	0.9391	0.9443
15,15	0.1	0.2838	0.2777	0.2612	0.3196	0.3210	0.3241	0.3454	0.3471	0.3499
	0.2	0.5974	0.5845	0.5780	0.6185	0.6211	0.6220	0.6437	0.6518	0.6547
	0.3	0.8536	0.8422	0.8327	0.8700	0.8710	0.8742	0.8890	0.8925	0.8945
	0.4	0.9664	0.9500	0.9453	0.9691	0.9729	0.9744	0.9710	0.9751	0.9763
20,10	0.1	0.2624	0.2551	0.2492	0.3130	0.3211	0.3232	0.3300	0.3347	0.3451
	0.2	0.5520	0.5422	0.5395	0.5912	0.6014	0.6088	0.6197	0.6229	0.6239
	0.3	0.8004	0.7816	0.7777	0.8212	0.8245	0.8265	0.8406	0.8436	0.8454
	0.4	0.9312	0.9253	0.9185	0.9560	0.9569	0.9645	0.9592	0.9634	0.9665
20,15	0.1	0.3001	0.2899	0.2774	0.3190	0.3222	0.3254	0.3454	0.3470	0.3536
	0.2	0.6282	0.6290	0.6179	0.6564	0.6599	0.6641	0.6697	0.6712	0.6754
	0.3	0.8688	0.8530	0.8412	0.8738	0.8755	0.8763	0.8957	0.8977	0.8997
	0.4	0.9770	0.9635	0.9527	0.9799	0.9810	0.9854	0.9835	0.9841	0.9844
20,20	0.1	0.3354	0.3075	0.2963	0.3569	0.3689	0.3699	0.3723	0.3814	0.3854
	0.2	0.6421	0.6345	0.6254	0.6732	0.6756	0.6785	0.6915	0.6987	0.7024
	0.3	0.8800	0.8712	0.8632	0.8912	0.8934	0.8966	0.9100	0.9124	0.9137
	0.4	0.9805	0.9754	0.9709	0.9856	0.9891	0.9898	0.9900	0.9935	0.9997

From the Tables 5-8, we have the following observations:

1. The proposed tests achieve 90% power at $\alpha = 0.10$ for sample sizes $n_1 = 15, n_2 = 10$ with $c, d \geq 1$ and shift $\Delta = 0.4$. Additionally, 95% power of the proposed test at $\alpha = 0.05$ is achieved for sample sizes $n_1 = 15, n_2 = 15$ with $c \geq 1, d \geq 2$ and shift $\Delta = 0.4$. Moreover, with the increase of sample size, power of the proposed test also increases.

2. The estimated level of significance is achieved with sample size $n_1 = 10$, $n_2 = 10$ with $c = d = 1$ for $\alpha = 0.10$. However for $\alpha = 0.05$, the estimated level of significance is achieved with $n_1 = 20, n_2 = 20$ with $c = 2, d = 1$.

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