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Generalized Lehmann Alternative Type II Family of Distributions and Their Applications

Jisha Varghese[1,*], Krishna E.[2,†] and K.K. Jose[3,‡]
[1,3]St.Thomas College, Mahatma Gandhi University, Kottayam, India
[2]St. Joseph's College for Women, Alappuzha, India

ABSTRACT

A new generalized family called Generalized Lehmann Alternative Type II (GLA2) family is introduced and studied in this paper. Special cases of this family using Uniform and Kumaraswamy distributions as base are developed and their statistical properties studied. Generalized Lehmann Alternative Type II Exponential (GLA2E) distribution is also developed and its statistical properties are obtained along with application. The new distribution is applied to a real data set to show the effectiveness of the distribution and it is verified that the new model is a better model than the existing exponential model and Marshall-Olkin extended exponential model. A detailed study on the record value theory associated with GLA2E distribution is conducted. Using the mean, variance and covariance of upper record values of the extended model, BLUE's of location and scale parameters are obtained and future records are predicted which has a number of practical uses. The 95% confidence interval for location and scale parameters are also computed. The result is applied to a real data set to validate the results. Entropy of record values is derived. This result will be

[*]srmonicash@gmail.com
[†]ekrishna48@gmail.com
[‡]kkjstc@gmail.com

useful in characterization of record values based on entropies and a quantification of information contained in each additional record value based on entropy measure.

Key words and Phrases: *Lehmann Alternative , Entropy, Hazard rate function, Kumaraswamy distribution, Marshall-Olkin distribution, Record value.*

1 Introduction

The properties and estimation methods for parameters of the exponentiated family of distributions have been studied by many authors, see Gupta and Kundu (2001a, 2001b, 2007), Pal et al. (2006), Nadarajah and Kotz (2006a) and Nadarajah et al. (2013). Tahir and Nadarajah (2015) discussed about Lehmann alternative type family of distributions. In the literature there exist two types of Lehmann alternative type family of distributions for obtaining the exponentiated family of distributions.

1.1 Lehmann Alternative 1 (LA1)

If F(x) is the cdf of the baseline distribution, then LA1 family of distributions is obtained by taking the β^{th} - power of F(x) so that

$$G(x) = (F(x))^{\beta}, \tag{1}$$

where $\beta > 0$ is a positive real parameter.The probability density function (pdf) corresponding to (1) is

$$g(x) = \beta f(x)(F(x))^{\beta-1}, \tag{2}$$

where $f(x) = \frac{d}{dx}F(x)$ denotes the pdf of F. For any lifetime random variable t, the survival (reliability) function (sf), $\overline{G}(t)$, the hazard (failure) rate function (hrf), h(t), the reversed hazard rate function (rhrf), r(t), and the cumulative hazard rate function (chrf), H(t), associated with (1) and (2) are $\overline{G}(t) = 1 - [F(t)]^{\beta}$, $h(t) = \beta f(t)[F(t)]^{\beta-1}[1-[F(t)]^{\beta}]^{-1}$, $r(t) = \beta f(t)[F(t)]^{-1}$ and $H(t) = -\log[1-[F(t)]^{\beta}]$.

1.2 Lehmann Alternative 2 (LA2)

If F(x) is the cdf and $\overline{F}(x) = 1 - F(x)$ is the sf of the baseline distribution, then the survival function of LA2 family of distributions is obtained by taking the β^{th} - power of $\overline{F}(x)$ so that

$$\overline{G}(x) = [\overline{F}(x)]^{\beta}, \tag{3}$$

where β is a positive real parameter. The LA2 cdf may also be written as

$$G(x) = 1 - [1 - F(x)]^{\beta}. \tag{4}$$

The pdf corresponding to (4) is

$$g(x) = \beta f(x)[1 - F(x)]^{\beta-1} \tag{5}$$

For any lifetime random variable t, the sf, hrf, rhrf and chrf associated with (3) and (4) are $\overline{G}(t) = [1 - F(t)]^{\beta}$, $h(t) = \beta f(t)[1 - F(t)]^{-1}$, $r(t) = \beta f(t)[1 - F(t)]^{\beta-1}\{1 - [1 - F(t)]\beta\}^{-1}$ and $H(t) = -\beta \log[1 - F(t)]$.
Nadarajah and Kotz (2006 a), Nadarajah (2006) and Rao et al. (2013) used the LA2 approach for introducing exponentiated Fréchet, exponentiated Gumbel and exponentiated log-logistic distributions. For more applications see Abd-Elfattah and Omima (2009), Abd-Elfattah et al. (2010), Rao et al. (2012, 2013), and Al-Nasser and Al-Omari (2013).

1.3 Marshall - Olkin Family of Distributions

Let X be a rv with a distribution function $F(x)$ and survival function $\overline{F}(x)$. By adding a new parameter, say δ, Marshall and Olkin (1997) introduced a new family of distributions namely Marshall - Olkin family of distributions with distribution function G(x) given by

$$G(x) = \frac{F(x)}{\delta + (1 - \delta)F(x)}, \quad x \in R \quad and \quad \delta > 0. \tag{6}$$

The corresponding survival function is

$$\bar{G}(x) = \frac{\delta\overline{F}(x)}{1 - (1 - \delta)\overline{F}(x)}, \quad x \in R \quad and \quad \delta > 0. \tag{7}$$

If $\delta = 1$, then G=F. If F has a density and hazard rate function, r_F, then by using the survival function $\overline{G}$, the density of G is given by

$$g(x;\delta) = \frac{\delta f(x)}{(1-(1-\delta)\overline{F}(x))^2}, \quad x \in R \quad and \quad \delta > 0 \tag{8}$$

and hazard rate function is

$$h(x;\delta) = \frac{r_F(x)}{1-(1-\delta)\overline{F}(x)}, \quad x \in R \quad and \quad \delta > 0.$$

Recently, many authors have developed various Marshall - Olkin distributions with respect to Gamma, Pareto, Weibull, Burr, Gumbel, Fréchet, Rayleigh, Kumaraswamy, Linear Exponential, Lomax and other distributions. For details, see Jose and Alice (2003, 2004 a, 2004 b), Ghitany et.al (2005, 2007), Jayakumar and Mathew (2008), Jose et.al (2010, 2011), Jose and Rani (2013), Krishna et al. (2013 a, 2013 b), Jose and Remya (2015).

2 Generalized Lehmann Alternative Type II Family of Distributions

Let X be a random variable with cumulative distribution function (cdf) F(x). The survival function (sf) and probability density function (pdf) of X are denoted by $\bar{F}(x) = 1 - F(x)$ and f(x) respectively. By Lehmann Alternative Type II exponentiated family discussed in section (1.2), we can take the cdf as

$$T(x) = 1 - (\bar{F}(x))^{\beta}. \tag{9}$$

The sf is

$$\bar{T}(x) = (\bar{F}(x))^{\beta}. \tag{10}$$

The corresponding pdf is

$$t(x) = \beta f(x)(\bar{F}(x))^{\beta-1}. \tag{11}$$

Marshall and Olkin (1997) introduced a new method of adding a parameter to a family of distributions to develop the Marshall-Olkin family which is discussed in section (1.3) and cumulative distribution function is given in (6).

Applying (9) in (6), we get the new family of distributions called Generalized Lehmann Alternative Type II family with parameters (δ, β) with cdf

$$G(x) = \frac{T(x)}{1 - \bar{\delta}\overline{T}(x)}.$$

On simplification, we get

$$G(x) = \frac{1 - (\overline{F}(x))^{\beta}}{1 - \bar{\delta}(\overline{F}(x))^{\beta}}. \tag{12}$$

The sf is

$$\overline{G}(x) = 1 - \frac{1 - (\overline{F}(x))^{\beta}}{1 - \bar{\delta}(\overline{F}(x))^{\beta}}.$$

On simplification, we get

$$\overline{G}(x) = \frac{\delta(\bar{F}(x))^{\beta}}{[1 - \bar{\delta}(\bar{F}(x))^{\beta}]}. \tag{13}$$

The corresponding pdf is

$$g(x) = \frac{\delta\beta f(x)(\bar{F}(x))^{\beta-1}}{[1 - \bar{\delta}(\bar{F}(x))^{\beta}]^{2}}. \tag{14}$$

The new family is referred to as GLA2 (δ, β).

The hazard rate function is $h(x) = \frac{g(x)}{\bar{G}(x)}$ and is obtained as

$$h(x) = \frac{\beta f(x)}{\bar{F}(x)[1 - \bar{\delta}(\bar{F}(x))^{\beta}]}. \tag{15}$$

2.1 Maximum Likelihood Estimation

Let $X_1, X_2, ...X_n$ be a random sample of size n from GLA2 family, then the likelihood function is

$$L = (\delta\beta)^{n} \frac{\prod_{i=1}^{n} f(x_i \left[\overline{F}(x_i)\right]^{\beta-1})}{\prod_{i=1}^{n} \left[1 - \bar{\delta}\left[\overline{F}(x_i)\right]^{\beta}\right]^{2}}.$$

The log likelihood function is given by

$$logL = n\log(\delta\beta) + \sum_{i=1}^{n}\log f(x_i) + (\beta-1)\sum_{i=1}^{n}\log\overline{F}(x_i) - 2\sum_{i=1}^{n}\log\left[1-\bar{\delta}(\overline{F}(x_i))^{\beta}\right].$$

The partial derivatives of the log likelihood with respect to δ and β are obtained as

$$\frac{\partial\log L}{\partial\delta} = \frac{n}{\delta} - 2\sum_{i=1}^{n}\frac{(\overline{F}(x_i))^{\beta}}{\left[1-\bar{\delta}(\overline{F}(x_i))^{\beta}\right]}$$

and

$$\frac{\partial\log L}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log\overline{F}(x_i) + 2\sum_{i=1}^{n}\frac{\bar{\delta}(\overline{F}(x_i))^{\beta}\log\overline{F}(x_i)}{\left[1-\bar{\delta}(\overline{F}(x_i))^{\beta}\right]}.$$

In order to estimate the parameters, we have to solve the normal equations

$$\frac{\partial\log L}{\partial\delta} = 0; \quad \frac{\partial\log L}{\partial\beta} = 0. \tag{16}$$

Since (16) cannot be solved analytically, numerical iteration technique is used to get a solution for the parameters δ and β. One may use the nlm package in R software to get the maximum likelihood estimator (MLE) for the parameters.

3 Some Special Generalized Lehmann Alternative Type II Models

In this section, we obtain some special GLA2 models using Uniform distribution and Kumaraswamy distribution. Also we derive their probability density function (pdf), cumulative density function (cdf) and quantile and the different shapes of density function and hazard rate function.

3.1 Generalized Lehmann Alternative Type II Uniform distribution

Let X follows Uniform distribution with parameter θ with pdf, cdf and survival function $g(x) = \frac{1}{\theta}$, $G(x) = \frac{x}{\theta}$ and $\bar{G}(x) = 1 - \frac{x}{\theta}$ respectively. If we apply the cdf,

survival function and pdf of Uniform distribution in the cdf, survival function and pdf of GLA2 family given in (12), (13) and (14), we get the cdf, survival function and pdf of the new distribution called Generalized Lehmann Alternative Type II Uniform distribution with parameters δ, β, θ and is denoted by GLA2U.
The cdf is

$$G(x) = \frac{1-(1-\frac{x}{\theta})^{\beta}}{1-\bar{\delta}(1-\frac{x}{\theta})^{\beta}} \qquad x, \delta, \beta, \theta > 0.$$

On simplification, we get

$$G(x) = \frac{\theta^{\beta}-(\theta-x)^{\beta}}{\theta^{\beta}-\bar{\delta}(\theta-x)^{\beta}} \qquad x, \delta, \beta, \theta > 0. \tag{17}$$

The corresponding sf $\overline{G}(x) = 1 - G(x)$ is

$$\overline{G}(x) = \frac{\delta(\theta-x)^{\beta}}{\theta^{\beta}-\bar{\delta}(\theta-x)^{\beta}} \qquad x, \delta, \beta, \theta > 0. \tag{18}$$

The corresponding pdf is

$$g(x) = \frac{\delta\beta\theta^{\beta}(\theta-x)^{\beta-1}}{[\theta^{\beta}-\bar{\delta}(\theta-x)^{\beta}]^{2}} \qquad x, \delta, \beta, \theta > 0. \tag{19}$$

The density plot for different values of the parameters are given in Fig 1

The hazard rate function is $h(x) = \frac{g(x)}{\overline{G}(x)}$ and is obtained as

$$h(x) = \frac{\beta\theta^{\beta}}{(\theta-x)[\theta^{\beta}-\bar{\delta}(\theta-x)^{\beta}]} \qquad x, \delta, \beta, \theta > 0.$$

The graph of h(x) is given in Fig 2. It gives J- shaped and bath tub shaped curves.

The u^{th} quantile of GLA2U distribution can be obtained by inverting $G(x) = u$ and is given by

$$x_u = \theta \left\{ 1 - \left[\frac{(u-1)}{u\bar{\delta}+1} \right]^{\frac{1}{\beta}} \right\}, \tag{20}$$

where $0 < u < 1$.

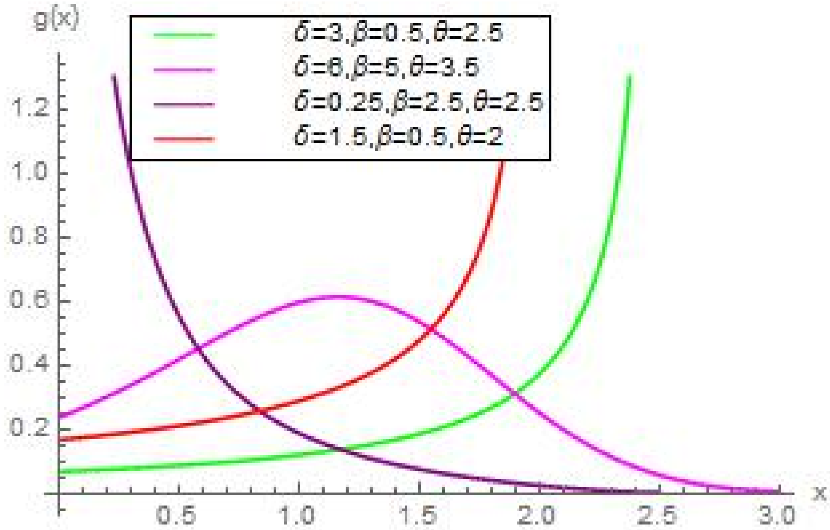

Figure 1: Probability density function of GLA2U for various values of δ, β and θ

3.2 Generalized Lehmann Alternative Type II Kumaraswamy Distribution

Let X follows Kumaraswamy distribution with parameters α and γ with cdf, survival function and pdf $F(x) = 1 - (1 - x^{\alpha})^{\gamma}$, $\bar{F}(x) = (1 - x^{\alpha})^{\gamma}$ and $f(x) = \alpha\gamma x^{(\alpha-1)}(1 - x^{\alpha})^{(\gamma-1)}$ respectively. If we apply the cdf, survival function and pdf of Kumaraswamy distribution in the cdf, survival function and pdf of Generalized Lehmann Alternative Type II family given in (12), (13) and (14), we get the cdf, survival function and pdf of the new distribution called Generalized Lehmann Alternative Type II Kumaraswamy distribution with parameters δ, β, α and γ is denoted by GLA2Kw.
The cdf is

$$G(x) = \frac{1 - (1 - x^{\alpha})^{\beta\gamma}}{1 - \bar{\delta}(1 - x^{\alpha})^{\beta\gamma}} \qquad x, \delta, \beta, \alpha, \gamma > 0. \tag{21}$$

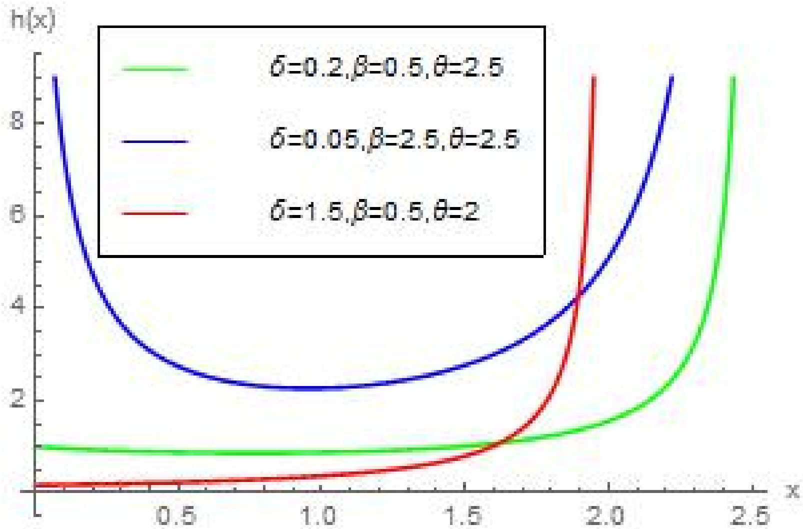

Figure 2: Hazard rate function of GLA2U for various values of δ, β and θ

The corresponding sf $\bar{G}(x) = 1 - G(x)$ is

$$\bar{G}(x) = \frac{\delta(1-x^{\alpha})^{\beta\gamma}}{1-\bar{\delta}(1-x^{\alpha})^{\beta\gamma}} \qquad x, \delta, \beta, \alpha, \gamma > 0. \tag{22}$$

The corresponding pdf is

$$g(x) = \frac{\delta\beta\alpha\gamma x^{(\alpha-1)}(1-x^{\alpha})^{(\gamma-1)}(1-x^{\alpha})^{\gamma(\beta-1)}}{[1-\bar{\delta}(1-x^{\alpha})^{\beta\gamma}]^2} \qquad x, \delta, \beta, \alpha, \gamma > 0. \tag{23}$$

The density plot for different values of the parameters are given in Figure 3.

The hazard rate function is $h(x) = \frac{g(x)}{\bar{G}(x)}$ and is obtained as

$$h(x) = \frac{\beta\alpha\gamma x^{(\alpha-1)}(1-x^{\alpha})^{(\gamma-1)}}{(1-x^{\alpha})^{\gamma}[1-\bar{\delta}(1-x^{\alpha})^{\beta\gamma}]} \qquad x, \delta, \beta, \alpha, \gamma > 0. \tag{24}$$

The plot of h(x) for different values of the parameters are given in Figure 4. It shows increasing and bath tub shaped curves.

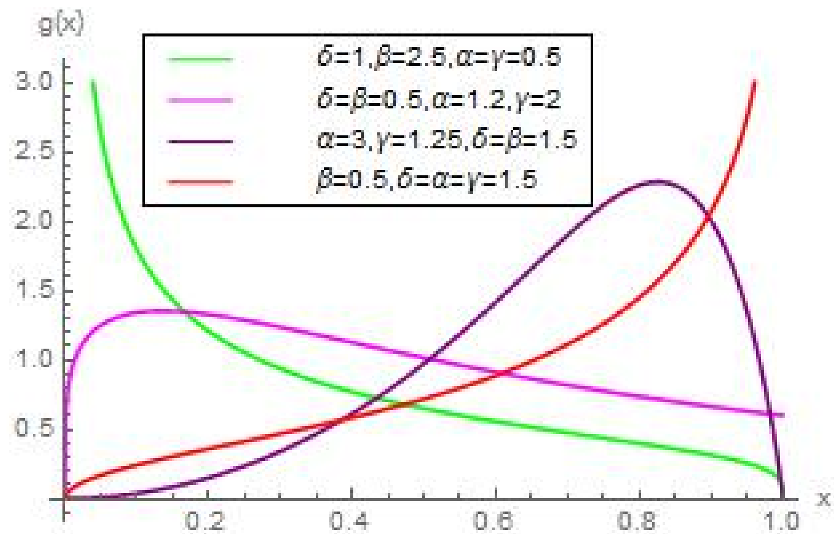

Figure 3: Probability density function of GLA2Kw for various values of δ, β, α and γ

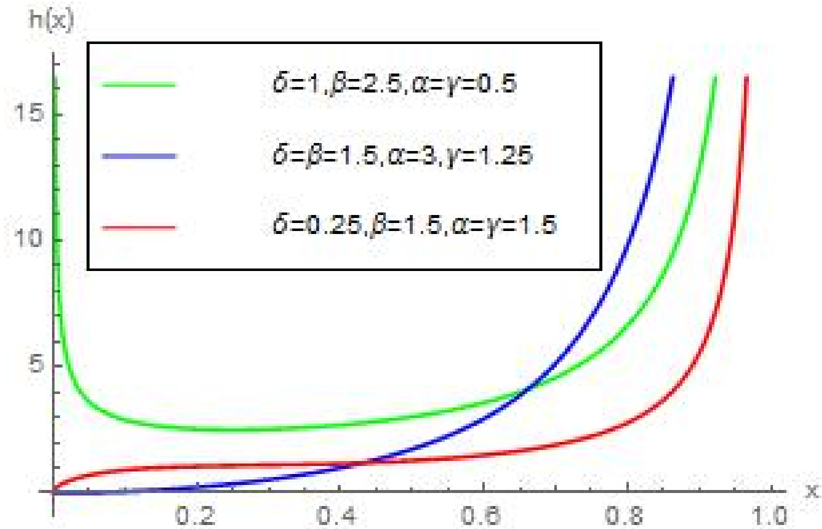

Figure 4: Hazard rate function of GLA2Kw for various values of δ, β, α and γ

The u^{th} quantile of GLA2Kw distribution can be obtained by inverting $G(x) = u$ and is given by

$$x_u = \left\{1 - \left[\frac{1-u}{1-u+u\delta}\right]^{\frac{1}{\beta\gamma}}\right\}^{\frac{1}{\alpha}} \tag{25}$$

where $0 < u < 1$.

3.3 Generalized Lehmann Alternative Type II Exponential Distribution

Exponential distribution plays a central role in analysis of lifetime or survival data, in part of their convenient statistical theory, their important lack of memory property and their constant hazard rates. In circumstances where the one-parameter family of exponential distributions is not sufficiently broad, a number of wider families such as the gamma, Weibull and Gompertz-Makeham distributions are in common use. Let $\overline{F}(x) = \mathrm{e}^{-\lambda x}, \quad x \geq 0$ is the survival function of exponential distribution, by

(9) we get the new distribution called Generalized Lehmann Alternative Type II Exponential (GLA2E) distribution with parameters (δ, β, λ) with cdf

$$G(x) = \frac{e^{\beta\lambda x} - 1}{e^{\beta\lambda x} - \overline{\delta}} \quad x \geq 0, \quad \lambda, \beta \ \ and \ \ \delta > 0, \overline{\delta} = 1 - \delta. \tag{26}$$

The survival function

$$\bar{G}(x) = \frac{\delta}{\mathrm{e}^{\beta\lambda x} - \overline{\delta}} \quad x \geq 0, \quad \lambda, \beta \ \ and \ \ \delta > 0, \overline{\delta} = 1 - \delta. \tag{27}$$

Then the pdf is

$$g(x) = \frac{\delta\beta\lambda \mathrm{e}^{\beta\lambda x}}{[\mathrm{e}^{\beta\lambda x} - \overline{\delta}]^2} \quad x \geq 0, \quad \lambda, \ \ \beta \ \ and \ \ \delta > 0, \overline{\delta} = 1 - \delta. \tag{28}$$

The graph of g(x) is given in Fig 5.

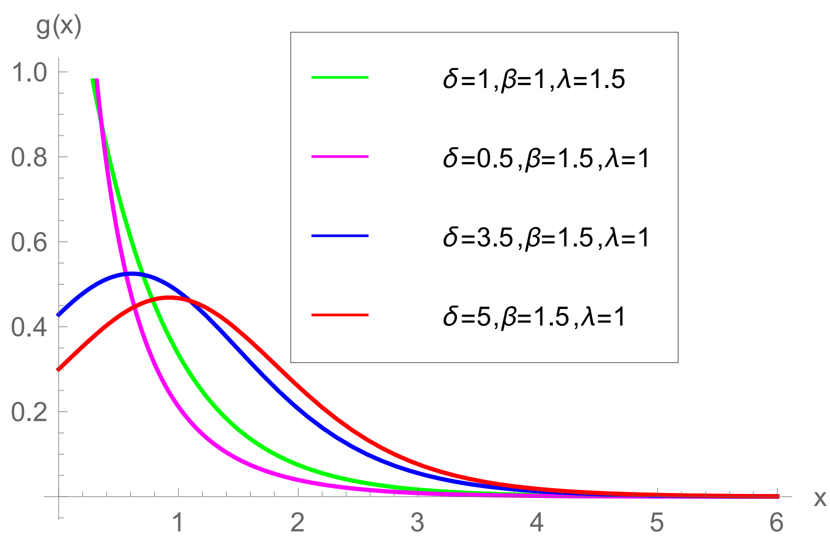

Figure 5: Probability density function of GLA2E (δ, β, λ) for various values of δ, β and λ

The hazard rate is

$$h(x) = \frac{\beta\lambda e^{\beta\lambda x}}{e^{\beta\lambda x} - \overline{\delta}} \quad x \geq 0, \quad \lambda, \ \ \beta \ \ and \ \ \delta > 0. \tag{29}$$

The graph of h(x) is given in Fig 6. It can be seen that the hazard rate is DFR for $\delta < 1$, and IFR for $\delta > 1$. Note that for $\delta = 1$, h(x)=1, showing constant failure rate. This establishes the wide applicability of the GLA2E distribution in reliability modeling.

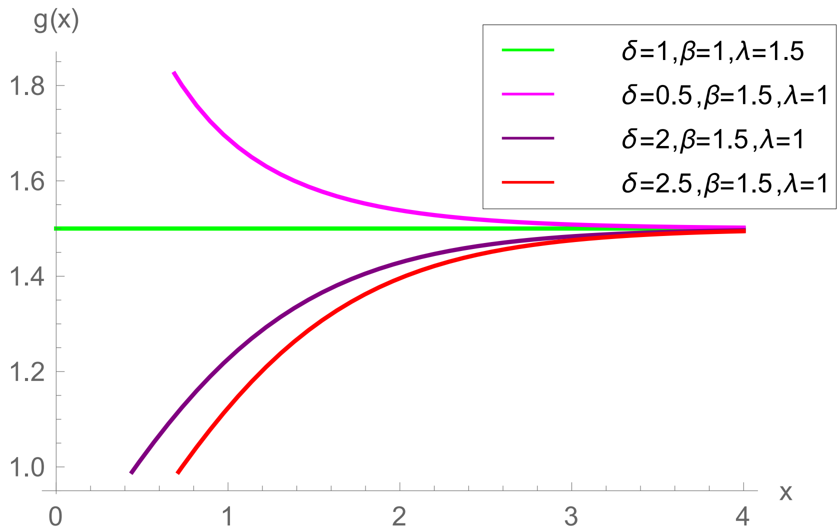

Figure 6: Hazard rate function of GLA2E (δ, β, λ) for various values of δ, β and λ

The u^{th} quantile is obtained by inverting the cdf given in (26).

$$x_u = \frac{1}{\beta\lambda} \log\left[\frac{u\bar{\delta} - 1}{u - 1}\right], \tag{30}$$

where U follows U(0,1).

3.3.1 Maximum Likelihood Estimation

Let $x_1, x_2, ..., x_n$ be a random sample of size n from $GLA2E(\delta, \beta, \lambda)$ distribution with pdf (28). The likelihood function is given by

$$L(\delta, \beta, \lambda) = \frac{(\delta\beta\lambda)^n e^{n\beta\lambda\bar{x}}}{\prod_{i=1}^{n}(e^{\beta\lambda x_i} - (1-\delta))^2}.$$

The log likelihood function is

$$logL = nlog(\delta\beta\lambda) + \sum_{i=1}^{n}\beta\lambda x_i - 2\sum_{i=1}^{n} log[e^{\beta\lambda x_i} - \overline{\delta}].$$

The MLE's of δ, β and λ are given by the solution of the three equations:

$$\frac{n}{\delta} - 2\sum_{i=1}^{n}\frac{1}{e^{\beta\lambda x_i} - \overline{\delta}} = 0, \tag{31}$$

$$\frac{n}{\lambda} + n\beta\overline{x} - 2\sum_{i=1}^{n}\frac{\beta x_i e^{\beta\lambda x_i}}{e^{\beta\lambda x_i} - \overline{\delta}} = 0 \tag{32}$$

and

$$\frac{n}{\beta} + n\lambda\overline{x} - 2\sum_{i=1}^{n}\frac{\lambda x_i e^{\beta\lambda x_i}}{e^{\beta\lambda x_i} - \overline{\delta}} = 0. \tag{33}$$

When $\delta = 1$, the model reduces to exponential distribution. Then we get, $\hat{\lambda} = \frac{1}{\overline{x}}$

Here we show that the Generalized Lehmann Alternative Type II model of Exponential distribution can be a better model than the one parameter exponential model and Marshall- Olkin Extended Exponential model when it is fitted for the following data. The data represents the failure times of the air conditioning system of an airplane reported in Linhart and Zucchini (1986) and is given in Table 1.

Using R program we estimate the parameters and obtain log likelihood, K-S statistic and p-value. The results are given in Table 2. From the table we can observe that the p-value is greater for GLA2E distribution than that of Exponential distribution and Marshall-Olkin Extended Exponential distribution. So we can

Table 1: Failure times of the air conditioning system of an airplane

23	261	87	7	120	14	62	47	225	71
246	21	42	20	5	12	120	11	3	14
71	11	14	11	16	90	1	16	52	95

Table 2: Summary statistics for the failure time data of the air conditioning system of an airplane.

Model	Parameter	MLE	-log L	K-S statistic	p-value
Exponential	λ	0.0168	152.6297	0.213	0.132
MOEE	δ	0.4072	151.425	0.129	0.6978
	λ	0.0106			
GLA2E	δ	0.3803			
	λ	0.0039	151.42	0.123	0.7508
	β	2.6135			

conclude that GLA2E distribution is a better model than Exponential distribution and Marshall-Olkin Extended distribution for the failure time data. The P-P plot and Q-Q plot for the data is given in Figure 7.

4 Record Value Theory for Generalized Lehmann Alternative Type II Exponential Distribution

Let $X_1, X_2, ...$ be an infinite sequence of i.i.d. random variables having the same distribution as the (population) random variable X. An observation X_j will be called an upper record value (or simply a record), if its value exceeds that of all previous observations. Then X_j is a record if $X_j > X_i$ for every $i < j$. The time at which records appear are of interest. Let X_j be observed at time j. Then the record time sequence $\{T_n,\ n \geq 0\}$ is defined as $T_0 = 1$ with probability 1 and for $n \geq 1, T_n = \min\{j : X_j > X_{T_{n-1}}\}$.The record value sequence $\{R_n\}$ is then defined by $R_n = X_{T_n}; \quad n = 1, 2, ...$ Then R_n is called the n^{th} record. Let $g_{R_n}(x)$ denote

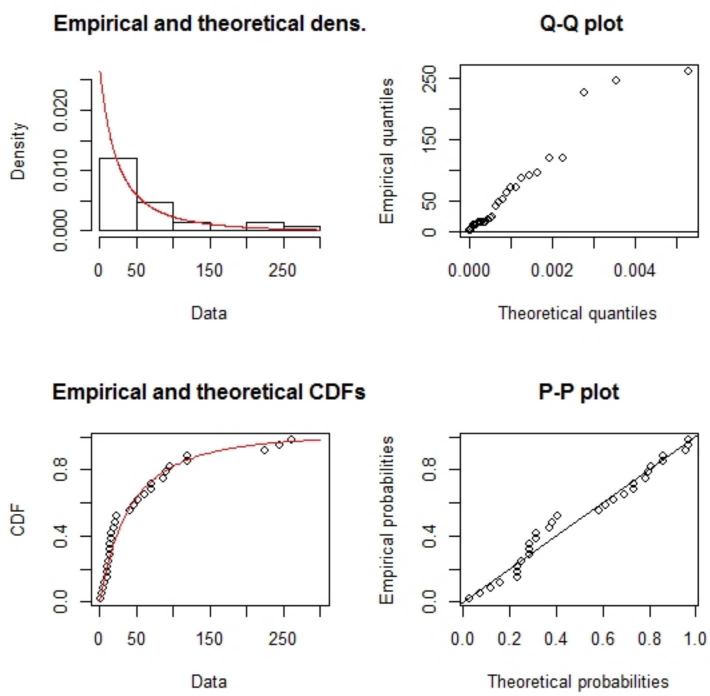

Figure 7: QQ plot and PP plot of GLA2E distribution

the p.d.f. of the n^{th} record. Then

$$g_{R_n}(x) = \frac{g(x)[-\log(\bar{G}(x))]^n}{n!}, -\infty < x < \infty. \tag{34}$$

The joint p.d.f. of a pair of records say R_m, R_n is given by

$$g_{R_m,R_n}(x,y) = \frac{[-\log \overline{G}(x)]^{m-1}}{(m-1)!} \frac{\left[-\log \frac{\overline{G}(y)}{\overline{G}(x)}\right]^{n-m-1}}{(n-m-1)!} \frac{g(x)g(y)}{1-G(x)}, -\infty < x < y < \infty. \tag{35}$$

Record data arise in a wide variety of practical situations such as industrial stress testing, meteorological analysis, hydrology, seismology, sporting and athletic events and oil mining surveys. In experiments related to these contexts measurements may be made sequentially and only the record values are observed. Usually the number of records of such experiments are considerably smaller than the complete sample size. This 'measurement saving' can be important when the measurements of these experiments are either costly or when the entire sample is destroyed.

Chandler (1952) introduced the study of record values and documented many of the basic properties of records. Arnold et al. (1998), Balakrishnan and Ahsanullah (1994), Balakrishnan et al. (1995), etc. have made significant contributions to the theory of records. Arnold et al. (1998) provide an excellent discussion on various results with respect to record values. Now we derive some record statistics with respect to Generalized Lehmann Alternative Type II Exponential distribution with $\lambda = 1$ for which the pdf is

$$g(x) = \frac{\delta\beta e^{\beta x}}{(e^{\beta x} - \overline{\delta})^2}, x > 0, \delta, \quad \beta > 0, \overline{\delta} = 1 - \overline{\delta} \tag{36}$$

By (34) the density function of the n^{th} record for $GLA2E(\delta, \beta, \lambda)$ distribution is given by

$$g_{R_n}(x) = \frac{\delta\beta \mathrm{e}^{\beta x}}{n![e^{\beta x} - \overline{\delta}]^2} \left[-\ln\left(\frac{\delta}{\mathrm{e}^{\beta x} - \overline{\delta}}\right)\right]^n, \qquad 0 < x < \infty \tag{37}$$

Then the single moment of n^{th} record statistic can be written as

$$\alpha_n = \frac{1}{\beta}\int_0^\infty ln(\overline{\delta} + \delta e^u)\frac{u^n}{n!}e^{-u}du. \tag{38}$$

Table 3: Mean of upper record values for $\beta = 1.5$

n	δ =1	δ =1.5	δ=2	δ =2.5	δ=3	δ =3.5	δ =4
1	1.3333	1.5434	1.7005	1.8265	1.9319	2.0225	2.1020
2	2	2.2410	2.4168	2.5554	2.6699	2.7675	2.8526
3	2.6667	2.9226	3.1038	3.2508	3.3691	3.4696	3.5568
4	3.3333	3.5965	3.7846	3.9311	4.0512	4.1528	4.2410
5	4	4.2668	4.4568	4.6044	4.7252	4.8275	4.9161
6	4.6667	4.9352	5.1261	5.2743	5.3955	5.4980	5.5869
7	5.3333	5.6028	5.7941	5.9426	6.0640	6.1666	6.2555

Theorem 4.1. *The single moment of n^{th} upper record value for $\delta > 0.5$ is given by*

$$\alpha_n = \frac{1}{\beta}\left\{\ln(\delta) + (n+1) - \sum_{i=1}^{\infty} \frac{k^i}{i(i+1)^{(n+1)}}\right\}, \quad where \quad k = 1 - \frac{1}{\delta}. \tag{39}$$

Proof From (38)and using the fact that $ln[1 - ke^{-u}] = -\sum_{i=1}^{\infty} \frac{k^i e^{-iu}}{i}$,

$$\alpha_n = \frac{1}{\beta}\left\{ln(\delta)\int_0^{\infty} \frac{u^n e^{-u}}{n!}du + \int_0^{\infty} \frac{u^{n+1}e^{-u}}{n!}du - \sum_{i=1}^{\infty} \frac{k^i}{i}\int_0^{\infty} \frac{e^{-(i+1)u}u^n}{n!}du\right\},$$

which on evaluation directly gives (39).

Using the result (39) the mean of record values from $GLA2E(\delta, \beta, \lambda)$ for different values of δ and for $\beta = 1.5$ and for $\delta = 1.5$ and for different values of β are evaluated and presented in Table 3 and Table 4.

Theorem 4.2. *The second single moment of n^{th} upper record value is*

$$\begin{aligned}\alpha_n^2 \quad = \quad & \frac{1}{\beta^2}\left\{\ln(\delta)^2 + (n+1)(n+2) + 2\ln(\delta) - 2(n+1)\sum_{i=1}^{\infty} \frac{k^i}{i(i+1)^{n+2}} - 2\ln(\delta)\right. \\ & \left. \times \sum_{i=1}^{\infty} \frac{k^i}{i(i+1)^{(n+1)}} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{k^{i+j}}{ij(i+j+1)^{(n+1)}}\right\}.\end{aligned} \tag{40}$$

Proof: From (38) the 2^{nd} single moment of n^{th} record value is given by

Table 4: Mean of upper record values for $\delta = 1.5$

n	β =1	β =1.5	β=2	β =2.5	β=3	β =3.5	β =4
1	2.3150	1.5434	1.1575	0.9260	0.7717	0.6614	0.5788
2	3.3615	2.2410	1.6808	1.3446	1.1205	0.9604	0.8404
3	4.3839	2.9226	2.1919	1.7536	1.4613	1.2525	1.0960
4	5.3948	3.5965	2.6974	2.1579	1.7983	1.5414	1.3487
5	6.4002	4.2668	3.2001	2.5601	2.1334	1.8286	1.6000
6	7.4028	5.1261	3.7014	2.9611	2.4676	2.1151	1.8507
7	8.4042	5.6028	4.2021	3.3617	2.8014	2.4012	2.1010

$$
\begin{aligned}
\alpha_n^2 &= \int_0^\infty \left\{\ln[\delta e^u(1-ke^{-u})]\right\}^2 \frac{u^n e^{-u}}{(n)!} du, \quad k = 1-\frac{1}{\delta} \\
&= (\ln\delta)^2 + (n+1)(n+2) + 2(n+1)\ ln\delta - 2(n+1)\sum_{i=1}^{\infty}\frac{k^i}{i(i+1)^{(n+2)}} - 2 \\
&\quad \times \ln\delta \sum_{i=1}^{\infty}\frac{k^i}{i(i+1)^{n+1}} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{k^{i+j}}{ij}\int_0^\infty e^{-(i+j+1)u}\frac{u^n}{(n)!}du.
\end{aligned}
$$

On simplification using the fact that $(a_1+a_2)^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} a_i a_j$ we get (40).

By (35) the joint pdf of m^{th} and n^{th} record values of GLA2E (δ, β) distribution is given by

$$
\begin{aligned}
g_{R_m,R_n}(x) &= \frac{\delta\beta^2\left[-\ln\left\{\dfrac{\delta}{\mathrm{e}^{\beta x}-(1-\delta)}\right\}\right]^m}{(m)!}\frac{1}{[\mathrm{e}^{\beta x}-(1-\delta)]}. \\
&\quad \times \frac{\left[-\ln\left\{\dfrac{\mathrm{e}^{\beta x}-(1-\delta)}{\mathrm{e}^{y}-(1-\delta)}\right\}\right]^{n-m-1}}{(n-m-1)!} \\
&\quad \times \frac{\mathrm{e}^{\beta y}}{[\mathrm{e}^{\beta y}-(1-\delta)]^2}, 0 < x < y < \infty.
\end{aligned}
$$

Theorem 4.3. *For $1 \leq m \leq n$ the product moment*

$$\begin{aligned}\alpha_{m,n} &= \frac{1}{\beta^2}\Bigg\{(\ln\delta)^2 + \ln\delta(m+n+2) + (m+1)(n+2) - [\ln\delta + (n-m)] \\ &\quad \times \sum_{i=1}^{\infty} \frac{k^i}{i(i+1)^{m+1}} - (m+1)\sum_{i=1}^{\infty} \frac{k^i}{i(i+1)^{m+2}} - \ln\delta \sum_{i=1}^{\infty} \frac{k^j}{j(j+1)^{n+1}} - (m+1) \\ &\quad \times \sum_{j=1}^{\infty} \frac{k^j}{j(j+1)^{n+2}} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{k^{(i+j)}}{ij(j+1)^{n-m}(i+j+1)^{(m+1)}}\Bigg\} \qquad (41)\end{aligned}$$

Proof:

$$\alpha_{m,n} = \frac{\delta\beta^2}{(m)!}\int_0^{\infty} x\left[-\ln\left(\frac{\delta}{e^{\beta x} - \overline{\delta}}\right)\right]^{m-1} \frac{e^{\beta x}}{e^{\beta x} - \overline{\delta}}\ I_x\ dx \qquad (42)$$

where

$$I_x = \frac{1}{(n-m-1)!}\int_x^{\infty} \frac{y e^{\beta y}}{(e^{\beta y} - \overline{\delta})^2}\left[-\ln\left(\frac{e^{\beta x} - \overline{\delta}}{e^{\beta y} - \overline{\delta}}\right)\right]^{(n-m-1)} dy$$

now making use of the transformation $u = -\ln\left(\frac{e^{\beta x} - \overline{\delta}}{e^{\beta y} - \overline{\delta}}\right)$

and writing $\ln\left[1 - \left(\frac{\delta - 1}{e^x - \overline{\delta}}\right)e^{-u}\right] = -\sum_{i=1}^{\infty}\left(\frac{\delta - 1}{e^x - \overline{\delta}}\right)^i \frac{e^{-iu}}{i}$ we get

$$I_x = \frac{1}{\beta^2(e^{\beta x} - \overline{\delta})}\left[\ln(e^{\beta x} - \overline{\delta}) + (n-m) - \sum_{i=1}^{\infty}\left(\frac{\delta - 1}{e^x - \overline{\delta}}\right)^i \frac{1}{i(i+1)^{n-m}}\right]$$

substituting the expression of I_x in (42) and using the transformation $t = -\ln\left(\frac{\delta}{e^{\beta x} - \overline{\delta}}\right)$ yields (41). Using (39), (40) and (41) numerical values of variance and covariance of upper record values are obtained by MATLAB program for $\beta = 1.5$ and various values of δ and is presented in Table 5.

4.1 Estimation of the location and scale parameters

In industry experiments, the number of measurements can be made lesser if the record values are observed instead of complete sample for estimation of parameters. There are also situations in which an observation is stored if it is a record value. This includes studies in meteorology, hydrology, seismology athletic events and mining.

Table 5: Variance and covariance of upper record values for $\beta = 1.5$

m	n	$\delta = 1$	$\delta = 1.5$	$\delta = 2$	$\delta = 2.5$	$\delta = 3$	$\delta = 3.5$	$\delta = 4$
1	1	0.8889	0.9747	1.0295	1.0682	1.0973	1.1200	1.1384
	2	0.8889	0.9516	0.9900	1.0162	1.0354	1.0502	1.0619
	3	0.8889	0.9406	0.9718	0.9929	1.0083	1.0200	1.0293
	4	0.8889	0.9353	0.9632	0.9820	0.9957	1.0062	1.0144
	5	0.8889	0.9327	0.9591	0.9768	0.9898	0.9996	1.0074
	6	0.8889	0.9315	0.9570	0.9743	0.9869	0.9965	1.0041
	7	0.8889	0.9308	0.9560	0.9731	0.9855	0.9949	1.0024
2	2	1.3333	1.3943	1.4298	1.4532	1.4699	1.4825	1.4923
	3	1.3333	1.3783	1.4039	1.4204	1.4321	1.4408	1.4475
	4	1.3333	1.3706	1.3916	1.4051	1.4146	1.4217	1.4271
	5	1.3333	1.3669	1.3857	1.3978	1.4063	1.4126	1.4175
	6	1.3333	1.3650	1.3828	1.3943	1.4023	1.4083	1.4129
	7	1.3333	1.3641	1.3814	1.3925	1.4003	1.4061	1.4106
3	3	1.7778	1.8171	1.8387	1.8524	1.8619	1.8689	1.8743
	4	1.7778	1.8070	1.8228	1.8327	1.8396	1.8446	1.8484
	5	1.7778	1.8020	1.8151	1.8233	1.8289	1.8330	1.8362
	6	1.7778	1.7996	1.8113	1.8187	1.8237	1.8274	1.8303
	7	1.7778	1.7984	1.8095	1.8164	1.8212	1.8247	1.8274
4	4	2.2222	2.2463	2.2590	2.2670	2.2724	2.2763	2.2793
	5	2.2222	2.2401	2.2496	2.2554	2.2594	2.2622	2.2644
	6	2.2222	2.2371	2.2450	2.2498	2.2531	2.2554	2.2572
	7	2.2222	2.2356	2.2427	2.2470	2.2500	2.2521	2.2537
5	5	2.6667	2.6809	2.6883	2.6928	2.6959	2.6981	2.6998
	6	2.6667	2.6773	2.6828	2.6862	2.6884	2.6901	2.6913
	7	2.6667	2.6755	2.6801	2.6829	2.6848	2.6861	2.6871
6	6	3.1111	3.1193	3.1236	3.1261	3.1279	3.1291	3.1300
	7	3.1111	3.1173	3.1204	3.1223	3.1236	3.1245	3.1252
7	7	3.5556	3.5602	3.5626	3.5640	3.5650	3.5657	3.5662

Recently much studies have been made on parametric and non parametric inferences based on record values. Raquab (2002) obtained inference for generalised exponential distribution based on record statistics. Soliman et al. made a comparison of Bayesian and non-Bayesian estimates using record statistics from Weibull model. Sultan et al. (2008) obtained the estimation from record values and predicted future records for gamma distribution. Sultan (2010) discussed different methods of estimation based on record values from inverse Weibull distribution.

Consider the general location-scale family of distributions with cdf $F(x, \mu, \sigma) = F(\frac{x-\mu}{\sigma})$ and pdf $f(x, \mu, \sigma) = \frac{1}{\sigma} f(\frac{x-\mu}{\sigma})$ and assume that the upper record values $R_1, R_2, ...R_n$ are available. Then **BLUE**'s of μ and σ are given respectively by, (see Balakrishnan and Cochen, 1991)

$$\mu^* = \frac{\alpha^T \boldsymbol{\Sigma}^{-1} \alpha \mathbf{1}^T \boldsymbol{\Sigma}^{-1} - \alpha^T \Sigma^{-1} \mathbf{1} \alpha^T \boldsymbol{\Sigma}^{-1}}{(\alpha^T \boldsymbol{\Sigma}^{-1} \alpha)(\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}) - (\alpha^T \boldsymbol{\Sigma}^{-1} \mathbf{1})^2} \mathbf{R} = \sum_{i=1}^{n} a_i R_i \tag{43}$$

$$\sigma^* = \frac{\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1} \alpha^T \boldsymbol{\Sigma}^{-1} - \mathbf{1}^T \boldsymbol{\Sigma}^{-1} \alpha \mathbf{1}^T \boldsymbol{\Sigma}^{-1}}{(\alpha^T \boldsymbol{\Sigma}^{-1} \alpha)(\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}) - (\alpha^T \boldsymbol{\Sigma}^{-1} \mathbf{1})^2} \mathbf{R} = \sum_{i=1}^{n} b_i R_i \tag{44}$$

where α denotes the column vector of the expected values of observed upper record values from the distribution F(x), $\boldsymbol{\Sigma}$ denotes the variance-covariance matrix of the record values from the distribution F(x), and $\mathbf{1}$ is a column vector of dimension n with all its entries as 1.

The three parameter Generalized Lehmann Alternative Type II exponential distribution has the probability density function given by

$$g(y) = \frac{\delta e^{\frac{(y-\mu)}{\sigma}}}{\sigma(e^{(\frac{(y-\mu)}{\sigma})} - \bar{\delta})^2}, y > \mu, \delta, \sigma > 0,$$

where δ, μ and σ are the shape,location and scale parameters respectively. By making use of means, variances and covariances presented in Table 3, Table 4, and Table 5, we calculate the coefficients of **BLUE**s a_i and b_i, i=1,2,...n for different values of shape parameter δ and n and presented in Table 6 and Table 7. It can be noted from these tables that $\sum_{i=1}^{n} a_i = 1$ and $\sum_{i=1}^{n} b_i = 0$

Table 6: Coefficients of the BLUE of μ for $\beta = 1.5$

n	$\delta = 1$	$\delta = 1.5$	$\delta = 2$	$\delta = 2.5$	$\delta = 3$	$\delta = 3.5$	$\delta = 4$
2	2.9999 -1.9999	3.2124 -2.2124	3.3740 -2.3740	3.5058 -2.5058	3.6178 -2.6178	3.7148 -2.7148	3.8004 -2.8004
3	2.0000 -0.0002 -0.9998	2.1061 0.0262 -1.1323	2.1921 0.0403 -1.2324	2.2525 0.0612 -1.3137	2.3077 0.0750 -1.3827	2.3562 0.0854 -1.4416	2.3990 0.0948 -1.4938
4	1.6667 -0.0001 0.0000 -0.6666	1.7318 0.0259 0.0087 -0.7664	1.7818 0.0410 0.0220 -0.8448	1.8228 0.0597 0.0187 -0.9012	1.8574 0.0722 0.0221 -0.9517	1.8880 0.0819 0.0252 -0.9951	1.9151 0.0908 0.0271 -1.0330
5	1.5000 -0.0001 0.0000 0.0000 -0.4999	1.5419 0.0260 0.0096 0.0030 -0.5805	1.5750 0.0414 0.0217 0.0026 -0.6407	1.6024 0.0587 0.0209 0.0066 -0.6886	1.6259 0.0707 0.0243 0.0079 -0.7288	1.6469 0.0800 0.0277 0.0080 -0.7626	1.6655 0.0884 0.0300 0.0089 -0.7928
6	1.4000 -0.0001 0.0000 0.0000 0.0001 -0.4000	1.4268 0.0259 0.0102 0.0036 0.0008 -0.4673	1.4488 0.0417 0.0215 0.0042 0.0013 -0.5175	1.4674 0.0582 0.0220 0.0079 0.0025 -0.5580	1.4837 0.0699 0.0255 0.0094 0.0023 -0.5908	1.4986 0.0788 0.0291 0.0097 0.0033 -0.6195	1.5118 0.0869 0.0316 0.0107 0.0034 -0.6444
7	1.3333 -0.0001 0.0000 0.0000 0.0001 -0.0001 -0.3332	1.3493 0.0257 0.0106 0.0041 0.0011 0.0014 -0.3922	1.3635 0.0418 0.0215 0.0051 0.0019 0.0009 -0.4347	1.3760 0.0576 0.0228 0.0087 0.0031 0.0009 -0.4691	1.3872 0.0691 0.0266 0.0103 0.0033 0.0010 -0.4975	1.3978 0.0779 0.0302 0.0109 0.0040 0.0006 -0.5214	1.4074 0.0858 0.0326 0.0121 0.0041 0.0005 -0.5425

Table 7: Coefficients for the BLUE of σ for $\beta = 1.5$

n	$\delta = 1$	$\delta = 1.5$	$\delta = 2$	$\delta = 2.5$	$\delta = 3$	$\delta = 3.5$	$\delta = 4$
2	-1.4999 1.4999	-1.3671 1.3671	-1.2942 1.2942	-1.2466 1.2466	-1.2126 1.2126	-1.1877 1.1877	-1.1678 1.1678
3	-1.4999 0.0003 0.4996	-1.3824 -0.0013 1.3837	-1.3162 0.0019 1.3143	-1.2729 -0.0058 1.2787	-1.2415 -0.0085 1.2500	-1.2181 -0.0103 1.2284	-1.1996 -0.0117 1.2113
4	-1.4999 0.0003 -0.0001 1.4997	-1.3897 -0.0014 0.0069 1.3842	-1.3269 0.0018 -0.0015 1.3266	-1.2850 -0.0064 0.0114 1.2800	-1.2546 -0.0090 0.0119 1.2517	-1.2319 -0.0109 0.0122 1.2306	-1.2137 -0.0127 0.0125 1.2139
5	-1.4999 0.0003 -0.0001 0.0001 1.4996	-1.3930 -0.0021 0.0072 0.0073 1.3806	-1.3321 0.0016 -0.0013 0.0157 1.3161	-1.2907 -0.0067 0.0111 0.0119 1.2744	-1.2609 -0.0094 0.0120 0.0128 1.2455	-1.2384 -0.0113 0.0121 0.0138 1.2238	-1.2203 -0.0132 0.0122 0.0140 1.2073
6	-1.4999 0.0003 -0.0001 0.0001 -0.0003 1.4999	-1.3946 -0.0022 0.0069 0.0074 0.0058 1.3767	-1.3343 0.0012 -0.0011 0.0154 0.0087 1.3101	-1.2935 -0.0071 0.0113 0.0119 0.0088 1.2686	-1.2638 -0.0098 0.0123 0.0125 0.0102 1.2386	-1.2415 -0.0118 0.0123 0.0138 0.0099 1.2173	-1.2236 -0.0134 0.0121 0.0141 0.0101 1.2007
7	-1.4999 0.0003 -0.0001 0.0001 -0.0003 0.0005 1.4994	-1.3957 -0.0019 0.0068 0.0074 0.0061 0.0023 1.3750	-1.3355 0.0012 -0.0014 0.0157 0.0084 0.0050 1.3066	-1.2951 -0.0069 0.0113 0.0119 0.0088 0.0060 1.2640	-1.2654 -0.0096 0.0119 0.0128 0.0097 0.0065 1.2341	-1.2430 -0.0116 0.0121 0.0135 0.0101 0.0071 1.2118	-1.2251 -0.0135 0.0121 0.0138 0.0104 0.0075 1.1948

The variances and covariance of the BLUE's of μ and σ are given by (see Balakrishnan and Cochen,1991)

$$Var(\mu^*) = \sigma^2 \left\{ \frac{\alpha^T \mathbf{\Sigma}^{-1} \alpha}{(\alpha^T \mathbf{\Sigma}^{-1} \alpha)(\mathbf{1}^T \mathbf{\Sigma}^{-1} \mathbf{1}) - (\alpha^T \mathbf{\Sigma}^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_1$$

$$Var(\sigma^*) = \sigma^2 \left\{ \frac{\mathbf{1}^T \mathbf{\Sigma}^{-1} \mathbf{1}}{(\alpha^T \mathbf{\Sigma}^{-1} \alpha)(\mathbf{1}^T \mathbf{\Sigma}^{-1} \mathbf{1}) - (\alpha^T \mathbf{\Sigma}^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_2$$

$$Cov(\mu^*, \sigma^*) = \sigma^2 \left\{ \frac{-\alpha^T \mathbf{\Sigma}^{-1} \mathbf{1}}{(\alpha^T \mathbf{\Sigma}^{-1} \alpha)(\mathbf{1}^T \mathbf{\Sigma}^{-1} \mathbf{1}) - (\alpha^T \mathbf{\Sigma}^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_3$$

Using these results Variance and covariances of the BLUE's of μ and σ can be obtained in terms of σ^2 and is presented in Table 8.

Example: Consider a simulated data of failure times which follow GLA2E distribution with $\alpha = \delta = 1.5$ *and* $\lambda = 1$,
1.3517, 1.8239, 0.1316, 1.18816, 0.8503, 0.1002, 0.3045, 0.6889, 2.3664, 2.4953, 0.1649, 2.6148, 2.3610, 0.5877, 1.2983, 0.1477, 0.4927, 1.9005, 1.2699, 2.3988, 0.9000, 0.0360, 1.4968, 2.0675, 0.9518, 1.1593, 1.1168, 0.4514, 0.8994, 0.1799, 1.0178, 0.0321, 0.3026, 0.0467, 0.997, 1.3860, 0.9900, 0.3524, 2.2593, 0.0348, 0.5173, 0.4368, 1.1830, 1.2803, 0.1975.
The observed upper record values are then,
1.3517, 1.8239, 1.8816, 2.3664, 2.4953, 2.6148.
With n=6, $\alpha = \delta = 1.5$ *and* $\lambda = 1$, the BLUE's of μ andσ can be computed using (43), (44) and Tables 6 and 7. The estimates are $\mu^* = 0.7837$ and $\sigma^* = 1.7557$
The corresponding variances and covariance of μ^* and σ^* can be obtained from Table 8
$Var(\mu^*) = 1.4757, Var(\sigma^*) = 0.9132$ and $Cov(\mu^* \sigma^*) = -0.3122$.

5 Confidence interval

Through the pivotal quantities

$$R_1 = \frac{\mu^* - \mu}{\sigma\sqrt{V_1}}, \quad R_2 = \frac{\sigma^* - \sigma}{\sigma\sqrt{V_2}} \quad \text{and} \quad R_3 = \frac{\mu^* - \mu}{\sigma^*\sqrt{V_1}}$$

Table 8: Variance and covariances of the BLUE's of μ and σ in terms of σ^2 for $\beta = 1.5$

n	$\delta = 1$	$\delta = 1.5$	$\delta = 2$	$\delta = 2.5$	$\delta = 3$	$\delta = 3.5$	$\delta = 4$
2	2.6662 0.9998 -1.3330	3.3570 0.9129 -1.5104	3.9183 0.8660 -1.6437	4.3993 0.8363 -1.7524	4.8230 0.8157 -1.8447	5.1994 0.8004 -1.5834	5.5422 0.7887 -1.9931
3	1.7777 0.9998 -0.6666	2.1912 0.9131 -0.7636	2.5267 0.8666 -0.8395	2.7977 0.8373 -0.8945	3.0384 0.8170 -0.9435	3.2512 0.8020 -0.9854	3.4433 0.7904 -1.0224
4	1.4814 0.9998 -0.4444	1.7968 0.9132 -0.5137	2.0436 0.8668 -0.5651	2.2490 0.8375 -0.6064	2.4255 0.8173 -0.6408	2.5804 0.8023 -0.6701	2.7196 0.7908 -0.6960
5	1.3333 0.9998 -0.3333	1.5968 0.9132 -0.3880	1.8000 0.8668 -0.4284	1.9676 0.8376 -0.4608	2.1103 0.8173 -0.4876	2.2351 0.8024 -0.5106	2.3464 0.7909 -0.5306
6	1.2444 0.9998 -0.2666	1.4757 0.9132 -0.3122	1.6517 0.8668 -0.3457	1.7952 0.8376 -0.3724	1.9170 0.8173 -0.3946	2.0228 0.8024 -0.4134	2.1167 0.7910 -0.4299
7	1.1852 0.9998 -0.2222	1.3939 0.9132 -0.2612	1.5512 0.8668 -0.2900	1.6784 0.8376 -0.3128	1.7857 0.8173 -0.3318	1.8786 0.8025 -0.3479	1.9607 0.7910 -0.3619

where μ^* and σ^* are the BLUE's of μ and σ we construct confidence interval for the location and scale parameters . We use R_1 and R_3 to construct CIs for μ when σ is known and when σ is unknown respectively while R_3 is used to construct CI's for σ. The construction of CI's require the percentage points of R_1, R_2 and R_3 which is obtained by using the BLUE's μ^* and σ^* via Monte carlo simulation based on 10000 runs and are presented in Table 9, 10 and 11 respectively.

6 Application

Now we apply the inference procedure discussed in the previous section to upper records of simulated data sets of size n=2,3,4,5,6 and 7 (with $\mu = 0, \sigma = 1$ and $\beta = 1.5 \; and \; \delta = 2$). The BLUE's are calculated using Tables 6 and 7 and is presented in Table 12

Using the BLUE's given in Table 12 and the percentage points of R_1 and R_3 we construct 95% confidence interval for μ when σ known and σ unknown respectively through the formulae,

$$P(\mu^* - \sigma\sqrt{V_1}R_1(97.5) \leq \mu \leq \mu^* - \sigma\sqrt{V_1}R_1(2.5)) = 95\%$$

$$P(\mu^* - \sigma^*\sqrt{V_1}R_3(97.5) \leq \mu \leq \mu^* - \sigma^*\sqrt{V_1}R_3(2.5)) = 95\%$$

We also construct confidence interval for σ using percentage points of R_2 through the formula

$$P(\frac{\sigma^*}{1+\sqrt{V_2}R_2(97.5)} \leq \sigma \leq \frac{\sigma^*}{1+\sqrt{V_2}R_2(2.5)}) = 95\%$$

The result is presented in table 13.

7 Prediction for Future Records

Prediction of future records becomes a problem of great interest. For example,while studying the record rainfall or snowfall,having observed the record values until the

Table 9: Simulated percentage points of R_1

α	n	0.5%	2.5%	5%	95%	97.5%	99.5%
1	2	-5.6048	-4.3647	-3.6886	1.0680	1.2553	1.6490
	3	-3.7185	-2.8144	-2.4643	0.6801	0.9140	1.3233
	4	-2.7703	-2.1807	-1.8877	0.4046	0.6224	0.9843
	5	-2.1528	-1.6737	-1.4879	0.2438	0.4602	0.7969
	6	-1.8005	-1.4185	-1.2617	0.1494	0.3165	0.6166
	7	-1.5893	-1.2325	-1.0969	0.0862	0.2277	0.5063
1.5	2	-5.6466	-4.3540	-3.7946	1.0855	1.2716	1.6391
	3	-3.7999	-2.9755	-2.6006	0.7355	0.9606	1.3155
	4	-2.8989	-2.2755	-2.0017	0.4559	0.6564	1.0353
	5	-2.3951	-1.8908	-1.6573	0.2899	0.4698	0.8471
	6	-1.9860	-1.5645	-1.3872	0.1933	0.3909	0.7205
	7	-1.7495	-1.3776	-1.2197	0.1200	0.2753	0.5728
2	2	-5.6107	-4.3902	-3.8306	1.0717	1.2635	1.5995
	3	-3.9064	-3.0607	-2.6715	0.7296	0.9569	1.3151
	4	-3.1229	-2.4339	-2.1276	0.5083	0.7113	1.1249
	5	-2.4654	-1.9295	-1.7106	0.3350	0.5625	0.9132
	6	-2.1414	-1.6892	-1.4868	0.2611	0.4199	0.7161
	7	-1.8242	-1.4775	-1.3091	0.1706	0.3485	0.6791
2.5	2	-5.8927	-4.4976	-3.8838	1.0709	1.2466	1.5387
	3	-4.0531	-3.0908	-2.7025	0.7249	0.9657	1.2860
	4	- 3.0235	-2.4615	-2.1567	0.5309	0.7412	1.1419
	5	-2.6150	-2.0163	-1.7838	0.3883	0.5817	0.9316
	6	-2.2246	-1.7543	-1.5669	0.2762	0.4315	0.7423
	7	-1.9673	-1.5583	-1.3835	0.1807	0.3479	0.6522
3	2	-5.8003	-4.4627	-3.9436	1.0356	1.2033	1.5224
	3	-4.1028	-3.2689	-2.8776	0.7613	0.9902	1.3394
	4	-3.1483	-2.5294	-2.2474	0.5344	0.7431	1.1134
	5	-2.6157	-2.0956	-1.8573	0.3815	0.5995	0.9595
	6	-2.3164	-1.8209	-1.5990	0.3059	0.4818	0.7932
	7	-1.9585	-1.5881	-1.4197	0.2162	0.3862	0.7111
3.5	2	-6.2370	-4.6718	-4.0112	1.0674	1.2328	1.5462
	3	-4.1166	-3.1950	-2.8594	0.7699	1.0098	1.3219
	4	-3.2041	-2.5486	-2.2757	0.5444	0.7572	1.1057
	5	-2.7375	-2.1398	-1.9060	0.3761	0.6090	0.9719
	6	-2.4066	-1.8791	-1.6463	0.3256	0.5062	0.8187
	7	-2.0798	-1.6544	-1.4697	0.2469	0.4224	0.7090
4	2	-6.0906	-4.6749	-4.0870	1.0676	1.2242	1.4689
	3	-4.2192	-3.2277	-2.8414	0.7766	0.9953	1.3071
	4	-3.2869	-2.6134	-2.3116	0.5683	0.7820	1.1489
	5	-2.7192	-2.1562	-1.9143	0.4332	0.6245	0.9703
	6	-2.3595	-1.9166	-1.7047	0.3404	0.5247	0.8608
	7	-2.1023	-1.6707	-1.4857	0.2233	0.4076	0.7151

Table 10: Simulated percentage points of R_2

α	n	0.5%	2.5%	5%	95%	97.5%	99.5%
1	2	-0.9838	-0.8978	-0.8094	3.9539	4.6854	6.2236
	3	-2.9830	-2.5203	-2.2619	0.5187	0.7560	1.3453
	4	-0.3644	0.0518	0.3188	4.7602	5.5493	7.2161
	5	-0.1531	0.3424	0.5932	4.8116	5.4976	7.0743
	6	0.1207	0.5766	0.8148	4.9689	5.5365	7.2471
	7	0.2645	0.6930	0.9076	5.0563	5.7481	7.4083
1.5	2	-1.0320	-0.9571	-0.8635	3.8172	4.5124	6.0503
	3	-0.7996	-0.4741	-0.2231	4.4816	5.1672	6.6805
	4	-0.4557	0.0145	0.2936	4.7694	5.4141	6.9187
	5	-0.1041	0.3609	0.6511	4.9347	5.6011	7.2818
	6	0.0580	0.5663	0.8464	4.9561	5.6505	7.0729
	7	0.3164	0.7543	1.0037	5. 0623	5.7650	7.2539
2	2	-1.0522	-0.9774	-0.8833	3.7763	4.4201	5.9488
	3	-0.8191	-0.4983	-0.2563	4.4266	5.1720	6.7478
	4	-0.4811	-0.0344	0.2837	4.7767	5.4815	7.1890
	5	-0.1934	0.3373	0.6073	4.8510	5.4855	6.9931
	6	0.1243	0.5631	0.8653	5.0114	5.6598	7.2408
	7	0.3080	0.7478	1.0209	5.0818	5.7963	7.2729
2.5	2	-1.0761	-1.0034	-0.9050	3.7554	4.4691	5.9813
	3	-0.8322	-0.4708	-0.2281	4.4342	5.0509	6.7658
	4	-0.4872	-0.0288	0.2877	4.7682	5.4150	6.7047
	5	-0.1494	0.3359	0.6389	4.8793	0.5533	7.1673
	6	0.1229	0.5822	0.8625	5.0375	5.6685	7.1661
	7	0.3350	0.8027	1.0718	5.1550	5.7967	7.2462
3	2	-1.0899	-1.0018	-0.9140	3.7177	4.3137	5.6924
	3	-0.8224	-0.5342	-0.2882	4.5343	5.2340	6.6902
	4	-0.4637	-0.0462	0.2718	4.8124	5.4531	6.8974
	5	-0.1437	0.3301	0.6146	4.9329	5.5515	6.8813
	6	0.1154	0.5954	0.8826	5.0871	5.7683	7.1481
	7	0.3127	0.8421	1.1048	5.1286	5.7708	7.2325
3.5	2	-1.1011	-1.0315	-0.9306	3.7736	4.4235	6.1671
	3	-0.8570	-0.5157	-0.2688	4.4775	5.0794	6.6519
	4	-0.4731	-0.0485	0.2204	4.7681	5.3593	6.8074
	5	-0.1657	0.3277	0.6398	4.9728	5.6169	7.1989
	6	0.1388	0.5945	0.9078	5.0823	5.7973	7.4037
	7	0.2979	0.8223	1.0760	5.2012	5.8126	7.2913
4	2	-1.1098	-1.0405	-0.9559	3.7920	4.4534	5.8867
	3	-0.8337	-0.5448	-0.3060	4.3638	5.0319	6.5454
	4	-0.5067	-0.0460	0.2424	4.7966	5.4323	6.8444
	5	-0.1765	0.3169	0.6245	4.9266	5.5686	6.9046
	6	0.0889	0.5909	0.8859	5.2025	5.8539	7.2088
	7	0.3425	0.8379	1.1365	5.1715	5.8387	7.3866

Table 11: Simulated percentage points of R_3

δ	n	0.5%	2.5%	5%	95%	97.5%	99.5%
1	2	-0.8146	-0.8057	-0.7950	5.2623	11.2520	77.3761
	3	-37.8845	-9.6445	-5.5812	3.5601	7.8030	47.7056
	4	-0.3645	-0.3616	-0.3579	0.2737	0.5256	1.3984
	5	-0.2881	-0.2860	-0.2830	0.1349	0.2979	0.7678
	6	-0.2387	-0.2369	-0.2346	0.0748	0.1703	0.4544
	7	-0.2038	-0.2024	-0.2007	0.0370	0.1095	0.3074
1.5	2	-0.8807	-0.8699	-0.8565	6.1249	13.1325	83.0720
	3	-0.5461	-0.5394	-0.5315	0.9181	1.5644	4.7084
	4	-0.4058	-0.3999	-0.3950	0.3298	0.6046	1.5776
	5	-0.3256	-0.3208	-0.3167	0.1638	0.3204	0.8235
	6	-0.2733	-0.2694	-0.2660	0.0971	0.2207	0.5773
	7	-0.2361	-0.2329	-0.2300	0.0541	0.1424	0.3666
2	2	-0.9223	-0.9085	-0.8927	5.9957	13.1293	60.6882
	3	-0.5813	-0.5706	-0.5608	0.9346	1.6818	4.6838
	4	-0.4348	-0.4260	-0.4183	0.3842	0.6907	1.8459
	5	-0.3518	-0.3445	-0.3390	0.1945	0.3876	0.9591
	6	-0.2979	-0.2919	-0.2872	0.1323	0.2514	0.5490
	7	-0.2581	-0.2535	-0.2494	0.0802	0.1820	0.4667
2.5	2	-0.9554	-0.9397	-0.9209	6.1792	13.6103	75.7308
	3	-0.6022	-0.5899	-0.5798	0.8775	1.6191	4.5777
	4	-0.4548	-0.4452	-0.4372	0.3880	0.7019	1.8229
	5	-0.3713	-0.3627	-0.3562	0.2331	0.4065	0.9663
	6	-0.3147	-0.3076	-0.3025	0.1400	0.2609	0.5943
	7	-0.2746	-0.2689	-0.2637	0.0834	0.1789	0.4137
3	2	-0.9783	-0.9606	-0.9432	5.8539	12.0516	76.3009
	3	-0.6211	-0.6079	-0.5967	1.0201	1.8443	4.7492
	4	-0.4718	-0.4613	-0.4512	0.4103	0.7234	1.6753
	5	-0.3866	-0.3764	-0.3692	0.2218	0.4409	1.0340
	6	-0.3283	-0.3202	-0.3138	0.1571	0.2892	0.6496
	7	-0.2876	-0.2798	-0.2747	0.0984	0.2001	0.4665
3.5	2	-0.9980	-0.9826	-0.9621	6.2831	14.5647	81.5251
	3	-0.6344	-0.6206	-0.6077	1.0010	1.7926	5.1915
	4	-0.4847	-0.4722	-0.4620	0.4342	0.7484	1.6421
	5	-0.3979	-0.3857	-0.3772	0.2309	0.4296	1.0322
	6	-0.3406	-0.3294	-0.3225	0.1691	0.3040	0.6460
	7	-0.2978	-0.2889	-0.2825	0.1156	0.2176	0.4955
4	2	-1.0122	-0.9956	-0.9728	6.8918	14.7036	88.2247
	3	-0.9834	-0.9309	-0.892	3.7529	5.8408	16.5728
	4	-0.4957	-0.4807	-0.4700	0.4514	0.7983	1.8950
	5	-0.4077	-0.3955	-0.3852	0.2694	0.4615	1.0219
	6	-0.3482	-0.3387	-0.3304	0.1817	0.3238	0.6890
	7	-0.3047	-0.2967	-0.2904	0.1033	0.2164	0.5076

Table 12: Upper Record values and BLUE's of μ and σ for$\beta = 1.5 and \delta = 2$

n	Upper record values	μ^*	σ^*
2	1.2920,2.7607	-2.1947	1.9008
3	1.0013,2.2525,2.6529	-0.9837	2.1731
4	1.0193,1.3935,2.1698,3.6395	-1.1536	3.4749
5	0.6795,1.2172,1.6170,3.6803,4.9475	-2.0046	5.6639
6	0.1976,0.2757,0.3639,1.4029,2.3866,2.8864	-1.1791	3.5601
7	1.1294,1.2049,1.4899,1.6223,2.9092,3.0884,3.1585	0.2659	2.6833

Table 13: 95% Confidence interval for μ and σ

n	2	3	4	5	6	7
95%CI forμ (σ known)	(-4.6958,6.4957)	(-2.5048,5.2259)	(-2.1704,2.3257)	(-2.7593,0.5840)	(-1.7188,0.9919)	(-0.1682, 2.1061)
95%CI forμ (σ unknown)	(-51.5954,1.2237)	(-6.7932,0.9874)	(-4.5846,0.9625)	(-4.9499,0.6131)	(-2.3294,0.1565)	(-0.3424,1.1285)
95%CI forσ	(0.3717,21.0192)	(0.3737,4.0533)	(0.5693,3.5899)	(0.9274,4.3103)	(0.5679,2.3356)	(0.4195,1.5819)

Table 14: Predicted records

n	simulated records of size (n-1)	Predicted value
3	2.3061,3.0550	3.5364
4	1.3610,1.4101,2.0967	4.1160
5	0.8188,1.2812,1.3946,2.2052	7.8762
6	0.3431,0.3987,0.9604,1.5604,3.1596	17.6655
7	0.9136,0.9301,1.1862,1.4686,2.2000,2.6492	13.3136
8	1.1294,1.2049,1.4899,1.6223,2.9092,3.0884,3.1585	17.6038

present time, we will be naturally interested in predicting the amount of rainfall or snowfall to be expected when the present record is broken for the first time in future. The best linear unbiased predicted value of the next record can be written as (see Balakrishnan and Chan, 1998).

$$y_{u(n)} = \mu^* + \sigma^* \beta_n$$

where μ^* andσ^* are the BLUE's based on the first (n-1) records and α_n is the n^{th} moment of record values. Prediction of next upper record value is obtained from a simulated data and presented in Table 14.

8 Entropy of Record value distribution

Entropy provides an excellent tool to quantify the amount of information (or uncertainty) contained in a random observation regarding its parent distribution. Shannon's(1948) entropy of an absolutely continuous random variable X with probability density function f(x) is given by

$$H_x[f(x)] = -\int_{-\infty}^{\infty} f(x) ln[f(x)] dx$$

The entropy is always non-negative in the case of a discrete random variable X and is also invariant under one-to- one transformation of X. For a continuous random variable, entropy is not invariant under one-to-one transformation of X and it takes values in $(-\infty, +\infty)$ .The entropy for some commonly used probability distributions have been tabulated by many authors. More recently Ebrahmi et al (2004) have explored the properties of entropy, Kullback - Leibler information and mutual information for order statistics. Now we discuss the entropy for the record values of GLA2E (δ, λ). Let$H_{(R_n)}$ be the entropy of the n^{th} record value. Then by Shakil (2005)

$$H_{(R_n)} = \ln(\Gamma n) - (n-1)\psi(n) - \frac{1}{\Gamma(n)} \int_{-\infty}^{\infty} [-\ln(1-G(x))]^{n-1} g(x) \ln(g(x)) dx \quad (45)$$

where $\int_0^\infty t^{j-1} e{-}tdt = \Gamma(j)$ and $\int_0^\infty t^{j-1} e^{-t} ln(t) dt = \Gamma(j)\psi(j)$ $\psi(j)$ is the digamma function.

For $n = 1$ entropy of the first record value is same as the entropy of parent distribution. Comparison of the entropy of parent distribution and n^{th} record value for $n \geq 2$ is same as comparison of entropy of first record value with entropy of a given n^{th} record value. Since the first observation from the parent distribution is always considered as a record value, entropy of the first non-trivial record value is obtained when $n \geq 2$.

Theorem 8.1. *For GLA2E (δ, β, λ) distribution if $H_{(j)}$ represents the entropy corresponding to j^{th} record, then*

$$H_{(j)} = \ln(\Gamma j) - (j-1)\psi(j) + j - ln(\sigma) + \sum_{i=1}^{\infty} \frac{k^i}{i(i+1)^j} \quad (46)$$

Proof By (45) the entropy of j^{th} record for GLA2E (δ, β, λ) is

$$H_{(j)} = \ln(\Gamma j) - (j-1)\psi(j) - \frac{1}{\Gamma(j)} \int_0^\infty \left[-\ln\left(\frac{\delta}{e^{\beta\lambda x} - \bar{\delta}} \right) \right]^{j-1} v(x) \ln v(x) \, dx$$

where $v(x) = \frac{\delta\lambda e^{\beta\lambda x}}{(e^{\beta\lambda x} - \bar{\lambda})^2}$. By the transformation $t = -\ln \frac{\delta}{e^{\beta\lambda x} - \bar{\delta}}$ and writing $\ln(1 - ke^{-t}) = -\sum_{i=1}^{\infty} \frac{k^i e^{-it}}{i}$ where $k = 1 - \frac{1}{\alpha}$ the result (42) can be easily obtained.

Table 15: Entropy of GLA2E (δ, β, λ)

Record	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$
2	1.6561	0.9630	0.2698	-0.1356
3	2.0250	1.3318	0.6387	0.2332
4	2.2538	1.5606	0.8675	0.4620
5	2.4118	1.7187	1.0255	0.6200
6	2.5296	1.8364	1.1433	0.7378
7	2.6226	1.9295	1.2363	0.8309

Using (46) the entropy of GLA2E (δ, β, λ) for $\delta = \beta = 1.5$ and for various record values and various values of λ are tabulated and presented in Table 15.

9 Conclusion

In this paper, a new family of distributions called Generalized Lehmann Alternative Type II family of distributions is introduced and explored the statistical properties such as probability density function (pdf), hazard rate function (hrf), expressions for cumulative distribution function (cdf), quantile and survival function. Maximum Likelihood function is obtained for estimation of unknown parameters of the new family of distributions. Different special models which include Uniform, Kumaraswamy models are developed for this new family. The probability density function (pdf), cumulative distribution function (cdf) and hazard rate function (hrf) are obtained and plotted the density function for different parameter values. A special model of this family called Generalized Lehmann Alternative Type II Exponential distribution is introduced and studied in detail. The statistical properties such as probability density function (pdf), hazard rate function (hrf), expressions for cumulative distribution function (cdf), quantile and survival function are obtained. The shapes of the density function and hazard rate function are plotted for different parameter values. Method of maximum likelihood estimation is used for estimation of unknown parameters of the new distribution. The new distribution is applied to a real data set to show the effectiveness of the distribution and it is verified that the new model is a better model than the existing exponential model and Marshall-Olkin extended exponential model. A detailed study on the record value theory associated with Generalized Lehmann Alternative Type II Exponential distribution is conducted. Using the mean, variance and covariance of upper record values of the extended model, BLUE's of location and scale parameters are obtained and future records are predicted which has a number of practical uses. The 95% confidence interval for location and scale parameters are also computed. MATLAB programs are developed for this purpose. The result is applied to a real data set to validate the results. Entropy of record values is derived. This result will be useful in characterization of record values based on entropies and a quantification of information contained in each additional record value based on entropy measure.

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