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Marshall-Olkin Extended Rayleigh Distribution and Applications

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ABSTRACT

In this paper a new lifetime distribution as a two parameter generalization of Rayleigh distribution namely Marshall Olkin Extended Rayleigh distribution is introduced and studied. The statistical properties such as quantile function, moments, compounding properties, Rényi entropy and stochastic ordering etc. are explored. The distribution of order statistics and its asymptotic distribution are derived. As an application of the distribution to the record value theory we evaluate the first and second single moment of n^{th} upper record value. The method of maximum likelihood is used to estimate the unknown parameters. The results are validated using real data sets.

Key words and Phrases: *Marshall-Olkin family, Rényi Entropy, Mean residual life time, Mean inactivity time, Order Statistics, Record values.*

1 Introduction

Statistical distributions have wide applications in modeling and analysis of real world phenomena. By considering the applicability, the theory of statistical distributions and new generalizations are developed. Several applications from engineering, financial, biomedical and other sectors show that the data often does not follow the

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classical distributions. It shows the need for extensions of distributions and through generalizations we can apply them in different areas. A common merit of the generalized distributions is that they have more parameters.

The methods of generalizations of the existing families of distributions for the derivation of more flexible distributions is available in the literature. For example Azzalini (1985), Marshall - Olkin (1997), Ferreira and Steel (2007), Aly and Benkherouf (2001) etc. introduced new methods for the extension of distributions. Based on the methods available in the literature several distributions were extended to new families of distributions such as skew normal distribution (Azzalini (1985)), Marshall-Olkin exponential distribution and Weibull families of distributions (Marshall-Olkin (1997)) etc.

Here we extended the Rayleigh distribution (Johnson et al (1994)) using the method of generalization given by Marshall-Olkin (1997) and its properties are studied. We discuss Marshall-Olkin extended Rayleigh distribution (MOER) with its applications.

This paper is organized as follows. Section 2 and 3 deals with the introduction of Marshall-Olkin (M-O) family of distributions and Rayleigh distribution. In Section 4 we derive the Marshall Olkin extended Rayleigh distribution along with the expansion of probability density function (pdf) and compounding property. Section 5 gives the statistical properties such as quantile function, moments, Rényi entropy, mean residual life function, mean inactivity time function and stochastic ordering. In Section 6 we study the application of the model in order statistics and record value theory. Section 7 deals with the estimation of parameters by using the maximum likelihood method of estimation and the estimates are validated with two real data sets. Section 8 gives the conclusion of the paper.

2 General Theory of Marshall-Olkin Family of Distributions

Motivated from reliability and survival contexts Marshall-Olkin (1997) proposed a general method of adding new parameter to an existing distribution. The corresponding family of distribution is known as Marshall-Olkin family of distributions.

This family contains the baseline distribution as its special case and give more flexibility to model various types of data. According to them if $\bar{F}(x)$ denote the survival or reliability function of a continuous random variable X, then the addition of a new parameter results in another survival function $\bar{G}(x)$ defined by

$$\bar{G}\left(x;\alpha\right)=\frac{\alpha\bar{F}\left(x\right)}{1-\bar{\alpha}\bar{F}\left(x\right)},\quad -\infty<x<\infty,\quad \alpha\geq 0,\ \bar{\alpha}=1-\alpha. \tag{2.1}$$

The M-O extended distribution offer a wide range of behavior than the basic distribution form which they are derived. When $\alpha=1$ in (2.1) we get $\bar{G}$=$\bar{F}$.
If F has the density f and hazard rate r_F then the corresponding density function and hazard rate function of the Marshall-Olkin family of distributions are given by

$$g\left(x;\alpha\right)=\frac{\alpha f\left(x\right)}{\left(1-\bar{\alpha}\bar{F}\left(x\right)\right)^{2}}, \tag{2.2}$$

$$h\left(x;\alpha\right)=\frac{r_F\left(x\right)}{1-\bar{\alpha}\bar{F}\left(x\right)}. \tag{2.3}$$

3 Rayleigh Distribution

The Rayleigh distribution is a special case of the Weibull distribution and applied in several areas including engineering, life testing and reliability which age with time as its hazard rate is a linear function of time. Several generalizations of Rayleigh distribution is given in the literature such as Generalized Rayleigh distribution introduced by Kundu and Raqab (2005), two parameter Burr Type X distribution is known as Generalized Rayleigh distribution is proposed by Surles and Padgett (2001) etc. Cordeiro et al (2013), Gomes et al (2014) and Jose and Sivdas (2015) generalizes the Rayleigh distribution as the Beta generalized Rayleigh distribution, four parameter Kumaraswamy generalized Rayleigh distribution and Negative binomial Marshall-Olkin Rayleigh distribution. Ozel and Cakmakyapan (2015) generalized the Rayleigh distribution based on the Marshall Olkin extension method. The acceptance sampling plan based on Inverse Rayleigh, Generalized Rayleigh and negative binomial Marshall- Olkin Rayleigh distributions are given in Rosaiah and Kantam (2005), Tsai and Wu (2006) and Jose and Sivdas (2015).

Rayleigh distribution is one of the most popular distributions for analyzing skewed data with applications in oceanography and in communication theory for describing instantaneous peak power areas such as health, agriculture, biology etc.

The cumulative distribution function (cdf) is given by

$$F(x) = 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right), \; x \geq 0, \; \sigma > 0 \tag{3.1}$$

The survival function of Rayleigh distribution is given as follows

$$\bar{F}(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right), \; x \geq 0, \; \sigma > 0 \tag{3.2}$$

The pdf of Rayleigh distribution is given by

$$f(x) = \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right), \; x \geq 0, \; \sigma > 0 \tag{3.3}$$

4 The Marshall-Olkin Extended Rayleigh Distribution

The survival function of the MOER (α, σ) can be obtained by substituting the survival function of Rayleigh distribution (3.2) in (2.1) and is given as follows

$$\bar{G}(x) = \frac{\alpha \exp\left(-\frac{x^2}{2\sigma^2}\right)}{\left(1 - \bar{\alpha}\exp\left(-\frac{x^2}{2\sigma^2}\right)\right)}, \quad x \geq 0, \; \alpha, \sigma > 0, \bar{\theta} = 1 - \theta \tag{4.1}$$

The pdf of (4.1) can be obtained directly form (2.2) as follows

$$g(x) = \frac{\alpha \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right)}{\left(1 - \bar{\alpha}\exp\left(-\frac{x^2}{2\sigma^2}\right)\right)^2}, \quad x \geq 0, \; \alpha, \sigma > 0 \tag{4.2}$$

The hazard rate function of MOER distribution given by

$$h(x) = \frac{\frac{x}{\sigma^2}}{\left(1 - \bar{\alpha}\exp\left(-\frac{x^2}{2\sigma^2}\right)\right)} \tag{4.3}$$

4.1 Compounding Property

Compounding of distributions gives a method for deriving new families of distributions in terms of the existing models. The Marshall Olkin generalized forms of several distributions were expressed as a compound distributions with exponential distribution as mixing density. Some of them are given in Ghitany et al (2005, 2007), Ghitany and Kotz (2007) and Krishana et al (2013). Let $\bar{G}(x/\theta), \; -\infty < x < \infty$, be the conditional survival function of a continuous random variable X given a continuous random variable Θ. Let Θ follows a distribution with the pdf $m(\theta)$. A distribution with the survival function

$$\bar{G}(x) = \int_{-\infty}^{\infty} \bar{G}(x/\theta) m(\theta)\, d\theta, \quad -\infty < x < \infty,$$

is called a compound distribution with mixing density $m(\theta)$. The following theorem shows that the MOER distribution can be expressed as a compound distribution.

Theorem 4.1. *Let the conditional survival function of a continuous random variable X given $\Theta = \theta$ be expressed as*

$$\bar{G}(x/\theta) = \exp\left(- \left(\exp\left(\frac{-x^2}{2\sigma^2} \right) \right)^{-1} - 1 \right) \theta, \ x > 0, \ \theta\,, \sigma > 0$$

Let Θ follows an exponential distribution with the pdf $m(\theta) = \exp(-\alpha\theta)$, $\theta > 0$, $\alpha > 0$. Then the compound distribution of X becomes the MOER (α, σ) distribution.

5 Statistical Properties

5.1 Quantile Function

The p^{th} quantile of MOER distribution is given by $x_p = G^{-1}(p)$

$$1 - \frac{\alpha \exp\left(\frac{-x^2}{2\sigma^2}\right)}{1 - \bar{\alpha} \exp\left(\frac{-x^2}{2\sigma^2}\right)} = p$$

on simplification we get the q^{th} quantile

$$x_q = \left(2\sigma^2 \log\left(\frac{1 - q\bar{\alpha}}{1 - q} \right) \right)^{\frac{1}{2}} \tag{5.1}$$

The median of the distribution can be obtained from the equation (5.1) as follows

$$\text{median} = \left(2\sigma^2 \log(\alpha + 1) \right)^{\frac{1}{2}}$$

5.2 Moments

Let X be random variable follows MOER (α, σ), then the kth moment of X is

$$\mu_k^{'} = E\left(x^k\right)$$

$$= \int_0^\infty x^k g\left(x\right)dx$$

By using (**??**) we can write

$$= \delta^* \int_0^\infty x^k \frac{\left(j+1\right)x}{\sigma^2} \exp\left(-\frac{\left(j+1\right)}{2\sigma^2}x^2\right)dx$$

If we put u=x^2,
On simplification we get

$$\mu_k^{'} = \delta^* \frac{\left(j+1\right)}{2\sigma^2} \frac{\Gamma\left(\frac{k}{2}+1\right)}{\left(\frac{1}{2\sigma^2}\left(j+1\right)\right)^{\frac{k}{2}+1}}$$

For k=1

$$\mu_1^{'} = \delta^* \sqrt{\frac{\pi}{2}} \frac{\sigma}{\left(j+1\right)}$$

For k=2

$$\mu_2^{'} = \delta^* \frac{\sigma^2}{\left(j+1\right)}$$

5.3 Rényi Entropy

The entropy of random variable X with density function $g(x)$ measures the variation of the uncertainty, which is used widely in studies related to science and engineering. The Rényi entropy of order δ is given as,

$$I_g\left(\delta\right) = \left(1-\delta\right)^{-1} \log \int_{-\infty}^{\infty} g^\delta\left(x\right)dx$$

where $\delta > 0$ and for $\delta \neq 1$ consider

$$\int_0^\infty g^\delta\left(x\right)dx = \frac{\alpha^\delta}{\left(\sigma^2\right)^\delta} \sum_{k=0}^{\infty} \bar{\alpha}^k \frac{\Gamma\left(2\delta+k\right)}{\Gamma\left(2\delta\right)\ k!} \int_0^\infty x^\delta \exp\left(\frac{-\left(\delta+k\right)x^2}{2\sigma^2}\right)dx$$

On simplification we have

$$I_g\left(\delta\right) \ = \left(1-\delta\right)^{-1}\log\frac{\alpha^{\delta}}{\left(\sigma^{2}\right)^{\delta}}\sum_{k=0}^{\infty}\bar{\alpha}^{k}\frac{\Gamma\left(2\delta+k\right)}{\Gamma\left(2\delta\right)\ k!}\frac{\Gamma\left(\frac{\delta+1}{2}\right)}{\left(\frac{\left(\delta+k\right)}{2\sigma^{2}}\right)^{\left(\frac{\delta+1}{2}\right)}}$$

For various values of $\delta = 3$ and for $\sigma = 0.5,\ 1.5$ and for various choices of α we can evaluate the Rényi entropy numerically as follows Table 1 displays the Rényi entropy for MOEP at $\sigma = 1.5,\ 1.2,\ \delta = 3$ and for different choices parameter α.

Table 1: Rényi entropy of MOER Distribution

α	σ	$I_g\left(\delta\right)\delta = 1.5$	$I_g\left(\delta\right)\delta = 2$	$I_g\left(\delta\right)\delta = 3$
0.1	1.5	2.2946	2.8991	1.9782
	1.2	1.0301	1.33710	0.8624
0.5	1.5	2.9103	3.4931	2.6106
	1.2	1.6458	1.9311	1.4928
1.0	1.5	3.0763	3.6522	2.7863
	1.2	1.81185	2.09018	1.6649
1.5	1.5	3.137	3.7115	2.8405
	1.2	1.8725	2.1495	1.7250
2.0	1.5	3.1639	3.7390	2.8674
	1.2	1.8995	2.1770	1.7531
2.5	1.5	3.1764	3.7524	2.8790
	1.2	1.9119	2.1904	1.7633

5.4 Mean Residual Life and Mean Inactivity Time

The mean residual life (MRL) function in reliability and survival analysis is useful because of its applications in describing the aging process. Let X be a random variable represents the life of a component, then the MRL is given by

$$m_r(t) = \frac{1}{\bar{G}\left(t\right)}\int_{t}^{\infty}\bar{G}\left(x\right)dx, \quad t > 0$$

Let X follows $MOER(\alpha,\,\sigma)$ then the MRL function of a lifetime random variable is given by

$$m_r(t)=\frac{\left(1-\bar{\alpha}\exp\left(-\frac{t^2}{2\sigma^2}\right)\right)}{\theta\exp\left(-\frac{t^2}{2\sigma^2}\right)}\int\limits_t^\infty\frac{\alpha\exp\left(-\frac{x^2}{2\sigma^2}\right)}{\left(1-\bar{\alpha}\exp\left(-\frac{x^2}{2\sigma^2}\right)\right)}dx$$

Table 2 displays the MRL function at $t=0.1,0.3,0.4$ for MOEP at $\sigma=1.5$, 0.5 and for different choices parameter α.

Table 2: Mean residual life of MOER Distribution

α	σ	MRL, t=0.1	MRL, t=0.3	MRL, t=0.4
0.1	1.5	0.3013	0.2351	0.2049
	0.5	1.0307	0.9041	0.8594
0.5	1.5	0.5742	0.3884	0.3064
	0.5	1.9207	1.7226	1.6252
1.0	1.5	0.7276	0.4928	0.3828
	0.5	2.4036	2.1830	2.0675
1.5	1.5	0.8246	0.5629	0.4361
	0.5	2.706	2.4739	2.3489
2.0	1.5	0.8957	0.6158	0.4773
	0.5	2.9270	2.6873	2.5561
2.5	1.5	0.9519	0.6583	0.5109
	0.5	3.1010	2.8558	2.7199

The mean inactivity time (MIT) function is also known as mean post lifetime and the mean waiting time function and has application in many disciplines such as reliability theory application in many disciplines such as reliability theory and survival analysis. (Kayid and Ahamad (2004)).

Let X be a random variable with distribution function G. Then MIT function of X is defined as follows.

$$m_I\left(t\right)=\frac{1}{G\left(t\right)}\int\limits_0^t G\left(x\right)dx,\ \ t>0$$

The MIT function of a lifetime random X with $MOER(\alpha, \sigma)$ distribution is

$$m_I\left(t\right) = \frac{\left(1-\bar{\alpha}\exp\left(-\frac{t^2}{2\sigma^2}\right)\right)}{1-\exp\left(-\frac{t^2}{2\sigma^2}\right)}\int\limits_0^t \frac{1-\exp\left(-\frac{x^2}{2\sigma^2}\right)}{\left(1-\bar{\alpha}\exp\left(-\frac{t^2}{2\sigma^2}\right)\right)}dx$$

Table 3 displays the MIT function at $t = 0.2, 0.5, 0.8$ for MOEP at $\sigma = 1.5,\ 0.5$ and for different choices parameter α.

Table 3: Mean Inactivity Time of MOER Distribution

α	σ	MIT, t=0.2	MIT, t=0.5	MIT, t=0.8
0.1	1.5	0.0763	0.2578	0.4958
	0.5	0.0679	0.1842	0.3289
0.5	1.5	0.0684	0.1911	0.3518
	0.5	0.0668	0.1697	0.2789
1.0	1.5	0.0672	0.1760	0.3054
	0.5	0.0667	0.1677	0.2709
1.5	1.5	0.0668	0.1703	0.2848
	0.5	0.0666	0.1670	0.2681
2.0	1.5	0.0665	0.1672	0.2731
	0.5	0.0666	0.1666	0.2667
2.5	1.5	0.0665	0.1654	0.2654
	0.5	0.0665	0.1664	0.2658

5.5 Stochastic Ordering

Stochastic ordering have been used in reliability theory, survival analysis, biological and clinical analysis, actuarial and economic theory etc. (Shaked and Shanthikumar (2007)). The concept of stochastic order is often useful in comparing random variables.

Theorem 5.1. *Let X and Y follows MOER (α_1, σ) and MOER (α_2, σ). If $\alpha_1 < \alpha_2$ then $X \leq_{lr} Y$ ($X \leq_{hr} Y$, $X \leq_{rhr} Y$, $X \leq_{st} Y$).*

Proof: let X follows MOER (α_1, σ) and let Y follows MOER(α_1, σ). If $\alpha_1 < \alpha_2$ then $X \leq_{hr} Y$ ($X \leq_{hr} Y$, $X \leq_{rhr} Y$, $X \leq_{st} Y$), consider

$$\frac{f(x)}{g(x)} = \frac{\alpha_1}{\alpha_2}\left(\frac{1-\alpha_2\exp\left(\frac{-x^2}{2\sigma^2}\right)}{1-\alpha_1\exp\left(\frac{-x^2}{2\sigma^2}\right)}\right)^2$$

since $\alpha_1 < \alpha_2$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = 2\frac{\alpha_1}{\alpha_2}(\alpha_1-\alpha_2)\left(\frac{x\exp\left(\frac{-x^2}{2\sigma^2}\right)\left(1-\alpha_2\exp\left(\frac{-x^2}{2\sigma^2}\right)\right)}{\left(1-\alpha_1\exp\left(\frac{-x^2}{2\sigma^2}\right)\right)^3}\right)$$
$$< 0$$

Hence $\frac{f(x)}{g(x)}$ is decreasing in x so that the random variable X is smaller than Y in the likelihood ratio order, that is $X \leq_{lr} Y$. The remaining statements follows from the implications given in (**??**). □

6 Order Statistics and Record value Theory

6.1 Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample taken from MOER distribution and $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be the corresponding order statistics. We derive the cdf of minimum order statistics $X_{1:n}$ which is denoted as $G_{1:n}(x)$, maximum order statistics $X_{n:n}$ which is denoted as $G_{n:n}(x)$, and pdf of the i^{th} order statistics which is denoted by $g_{i:n}(x)$. The general formula for the cdf of the first order statistics is given by

$$G_{1:n}(x) = 1-(1-G(x))^n$$
$$= 1-\alpha^n\sum_{i=0}^{\infty}\frac{\Gamma(n+i)}{\Gamma(n)\,i!}(1-\alpha)^i\left(\exp\left(-\frac{x^2}{2\sigma^2}\right)\right)^{n+i}$$

The cdf of the maximum order statistics $X_{n:n}$ for MOER distribution can be derived as

$$G_{n:n}(x) = (G(x))^n$$
$$= \left(1-\frac{\alpha\exp\left(\frac{-x^2}{2\sigma^2}\right)}{1-(1-\alpha)\exp\left(\frac{-x^2}{2\sigma^2}\right)}\right)^n$$
$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{n}{j}(-1)^j\frac{\Gamma(j+k)}{\Gamma(j)\,k!}\alpha^j(1-\alpha)^k\left(\exp\left(\frac{-x^2}{2\sigma^2}\right)\right)^{j+k}$$

The pdf of the i^{th} order statistics $X_{i:n}$ is given by

$$g_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} g(x) \left(\bar{G}(x)\right)^{n-i} \left(1-\bar{G}(x)\right)^{i-1}$$

$$= \frac{n!}{(i-1)!\,(n-i)!} g(x) \sum_{j=0}^{i-1} (-1)^i \left(\begin{array}{c} i-1 \\ j \end{array} \right) \left(\bar{G}(x)\right)^{n+i-j}$$

Substituting the pdf and cdf of MOER distribution and after simplification we get

$$g_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} \frac{\alpha \frac{x}{\sigma^2} \exp\left(\frac{-x^2}{2\sigma^2}\right)}{\left(1-\bar{\alpha}\exp\left(\frac{-x^2}{2\sigma^2}\right)\right)^2} \sum_{j=0}^{i-1} (-1)^i \left(\begin{array}{c} i-1 \\ j \end{array} \right) \left(\frac{\alpha \exp\left(\frac{-x^2}{2\sigma^2}\right)}{1-\bar{\alpha}\exp\left(\frac{-x^2}{2\sigma^2}\right)} \right)^{n+i-j}$$

Limiting Distributions of Sample Extremes

Let $X_1, X_2, ..., X_n$ be a random sample of size n from an absolutely continuous distribution with pdf $g(x)$ and cdf $G(x)$. Limiting distributions of sample minima $X_{1:n}{=}min\left(X_1, X_2, ..., X_n\right)$ and maxima $X_{n:n}{=}max\left(X_1, X_2, ..., X_n\right)$ can be derived by using the asymptotic results for $X_{1:n}$ and $X_{n:n}$ given in Arnold et al (1992) and Kotz and Nadarajah (2001).

For the minimum order statistics $X_{1:n}$ we have

$$\lim_{n\to\infty} P\left\{X_{1:n} \leq a_n^* + b_n^* x\right\} = 1 - \exp\left(-x^c\right), \quad x > 0,\ c > 0 \tag{6.1}$$

of Weibull type, where

$$a_n^* = G^{-1}(0) \text{ and } b_n^* = G^{-1}\left(\tfrac{1}{n}\right) - G^{-1}(0)$$

if and only if $G^{-1}(0)$ is finite and for all $x > 0$ and $c > 0$

$$\lim_{\in\to 0^+} \frac{G\left(G^{-1}(0) + \in t\right)}{G\left(G^{-1}(0) + \in\right)} = x^c \tag{6.2}$$

For the maximum order statistics $X_{n:n}$ we have

$$\lim_{n\to\infty} P\left\{X_{n:n} \leq a_n + b_n x\right\} = \exp\left(-\exp\left(-t\right)\right), \ -\infty < x < \infty \tag{6.3}$$

of Extreme value type, where $a_n = G^{-1}\left(1 - \frac{1}{n}\right)$ and $b_n = \left\{nf\left(a_n\right)\right\}^{-1}$ if

$$\lim_{x\to G^{-1}(1)} \frac{d}{dx}\left(\frac{1}{h(x)}\right) = 0 \tag{6.4}$$

The following theorem gives the limiting distributions of the smallest and largest order statistics from MOER distribution.

Theorem 6.1. *Let $X_{1:n}$ and $X_{n:n}$ be representing the smallest and largest order statistics from $MOER(\alpha, \sigma)$ distribution. Then the*

1. $\lim_{n\to\infty} P\left\{X_{1:n} \le a_n^* + b_n^* x\right\} = 1 - \exp\left(-x^c\right), \quad x > 0,\ c > 0$

$$a_n^* = G^{-1}(0) \text{ and } b_n^* = G^{-1}\left(\tfrac{1}{n}\right) - G^{-1}(0)$$

   and $G^{-1}(.)$ given by the quantile function given in (5.1).

2. $\lim_{n\to\infty} P\left\{X_{n:n} \le a_n + b_n x\right\} = \exp\left(-\exp\left(-t\right)\right), \ -\infty < x < \infty$ *where* $a_n = G^{-1}\left(1 - \frac{1}{n}\right)$ *and* $b_n = \left\{nf\left(a_n\right)\right\}^{-1}$. *Here $G^{-1}(.)$ and $g(.)$ given by (5.1) and (4.2) respectively.*

Proof:

1. For MOER distribution $G^{-1}(0) = 0$ is finite,
   By using L Hospitals rule, we have,

$$\lim_{\in\to 0^+} \frac{G\left(G^{-1}(0) + \in t\right)}{G\left(G^{-1}(0) + \in\right)} = t \lim_{\in\to 0^+} \frac{g(\in t)}{g(t)} = t$$

   Hence statement 1 follows from (6.1) and (6.2).

2. For the MOER distribution, we have

$$\lim_{x\to\infty} \frac{d}{dx}\left(\frac{1}{h(x)}\right)$$

$$= \lim_{x\to\infty}\left(\bar{\alpha}\exp\left(\frac{-x^2}{2\sigma^2}\right) - \frac{\left(1 - \bar{\alpha}\exp\left(\frac{-x^2}{2\sigma^2}\right)\right)}{x^2}\right)\sigma^2 = 0$$

   Hence the statement 2 follows from equations (6.3) and (6.4).

□

6.2 Record Value Theory

The theory of record values and many of the basic properties of records were introduced by Chandler (1952). Record data arises in a wide variety of practical situations such as industrial stress testing, meteorological analysis, hydrology, seismology, sporting and athletic events and oil and mining surveys. Arnold and Balakrishnan

(1989), Balakrishnanan and Ahsanullah (1994), Ahsanullah (1995, 1988), Sultan et al (2003) etc, have made significant contributions to the record value theory. Also characterizations based on entropy of record values is discussed by Baratpour et al (2008), Ahmadi (2009), Ahmadi and Fashandi (2009) etc. Arnold et al (1998) obtained various results with record values. Here we derive some results with record values.

Here we derive some of the record statistics with respect to MOER distribution given in (4.1) when $\sigma = 1$. Then the pdf corresponding to (4.2) is

$$g(x) = \frac{\alpha x \exp\left(x^2\right)}{\left(\exp\left(x^2\right) - \bar{\alpha}\right)^2}, \ \ x > 0, \ \alpha > 0, \ \bar{\alpha} = 1 - \alpha \tag{6.5}$$

Let $X_1, X_2, ...$ be an infinite sequence of independently and identically distributed random variables having the same distribution as the population random variable X. An observation X_j will be called an upper record value (or record) if its value exceeds that of all previous observations. Then X_j is a record if $X_j > X_i$ for every $i < j$. The time at which records occur are of interest. Let X_j be observed at time j, then the record time sequence $\{T_n, n \geq 0\}$ is defined as $T_0 = 1$ with probability 1 and for $n \geq 1, \quad T_n = Min\{j, X_j > X_{T_{n-1}}\}$.

The record value sequence $\{R_n,\, n \geq 1\}$ is defined by $R_n = \{X_{T_n}\}$, n=1,2,... . Let R_n denote the n^{th} record (Arnold et al (1998)), then the pdf of n^{th} record say $g_{R_n}(x)$ is given by

$$g_{R_n}(x) = \frac{g(x)\left(-\log\left(1 - G(x)\right)\right)^{n-1}}{(n-1)!}, \ -\infty < x < \infty \tag{6.6}$$

The joint pdf of a pair of record say $R_m,\, R_n$ is given by

$$g_{R_m,R_n}(x,y) = \frac{\left(-\log \bar{G}(x)\right)^{m-1}}{(m-1)!} \frac{\left(-\log\left(\frac{\bar{G}(y)}{\bar{G}(x)}\right)\right)^{n-m-1}}{(n-m-1)!} \frac{g(x)\, g(y)}{1 - G(x)}, \ \ -\infty < x < y < \infty \tag{6.7}$$

By using (6.6) and (6.5), the density function of the n^{th} record for $MOER(\alpha)$ distribution is given by

$$g_{R_n}(x) = \frac{\alpha x \exp\left(x^2\right)}{\left(\exp\left(x^2\right) - \bar{\alpha}\right)^2 (n-1)!} \left(-\ln\left(\frac{\alpha}{\left(\exp\left(x^2 - \bar{\alpha}\right)\right)}\right)\right)^{n-1}, \ 0 < x < \infty \tag{6.8}$$

The single moment of n^{th} upper record statistics can be written as

$$\beta_n = \int_0^\infty x^2 \frac{\alpha \exp\left(x^2\right)}{\left(\exp\left(x^2\right) - \bar{\alpha}\right)^2 (n-1)!} \left(-\ln\left(\frac{\alpha}{\left(\exp\left(x^2 - \bar{\alpha}\right)\right)}\right)\right)^{n-1} dx$$

Let $u = -\ln\left(\frac{\alpha}{(\exp(x^2 - \bar{\alpha}))}\right)$,
On simplification we get

$$\beta_n = \int_0^\infty \frac{\left(\ln\left(\alpha \exp\left(u\right) + \ln\left(1 - k\exp\left(-u\right)\right)\right)\right)^{\frac{1}{2}}}{2} \frac{u^{n-1}}{(n-1)!} \exp\left(-u\right) du$$

where $k = 1 - \frac{1}{\alpha}$.

The second single moment of n^{th} upper record value is

$$\beta_n^2 = \int_0^\infty x^2 \frac{\alpha x \exp\left(x^2\right)}{\left(\exp\left(x^2\right) - \bar{\alpha}\right)^2 (n-1)!} \left(-\ln\left(\frac{\alpha}{\left(\exp\left(x^2 - \bar{\alpha}\right)\right)}\right)\right)^{n-1} dx$$

Take $u = -\ln\left(\frac{\alpha}{(\exp(x^2 - \bar{\alpha}))}\right)$

$$\beta_n^2 = \int_0^\infty \frac{\left(\ln\left(\alpha \exp\left(u\right)\right) + \ln\left(1 - k\exp\left(-u\right)\right)\right)}{2} \frac{u^{n-1}\exp\left(-u\right)}{(n-1)!} du$$

where $k = 1 - \frac{1}{\alpha}$.

The joint pdf of m^{th} and n^{th} record values of $MOER(\alpha)$ distribution can be derived by using (6.7) and (6.5) and is given by

$$g_{R_m R_n}(x, y) = \alpha \frac{\left(-\ln\left(\frac{\alpha}{\exp(x^2 - \bar{\alpha})}\right)\right)^{m-1}}{(m-1)!} \frac{\left(-\ln\left(\frac{\exp\left(x^2 - \bar{\alpha}\right)}{\exp(y^2 - \bar{\alpha})}\right)\right)^{n-m-1}}{(n-m-1)!}$$
$$\frac{\exp\left(x^2\right)}{\exp\left(x^2 - \bar{\alpha}\right)} \frac{\exp\left(y^2\right)}{\left(\exp\left(y^2 - \bar{\alpha}\right)\right)^2}, \quad 0 < x < y < \infty \qquad (6.9)$$

7 Maximum Likelihood Estimators

In this section estimates of the unknown parameters of MOER distribution are derived by using the method of maximum likelihood based on a complete sample.

Let X_1, X_2,...,X_n be a random sample of size n from MOER distribution with pdf in (4.2). The maximum likelihood estimators (mle) of MOER distribution to estimate the model parameters are considered. The corresponding log likelihood function can be written as

$$LogL = n\log\alpha + \sum_{i=1}^{n}\log\frac{x_i}{\sigma^2} - \sum_{i=1}^{n}\frac{x_i}{2\sigma^2} - 2\sum_{i=1}^{n}\log\left(1-\bar{\alpha}\exp\left(\frac{-x_i^2}{2\sigma^2}\right)\right)$$

Differentiating $LogL$ with respect to each parameters and setting the results to zero we get the mle.

$$\frac{\partial LogL}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\frac{2\exp\left(\frac{-x_i{}^2}{2\sigma^2}\right)}{1-\bar{\alpha}\exp\left(\frac{-x_i{}^2}{2\sigma^2}\right)}$$

$$\frac{\partial LogL}{\partial\sigma} = \sum_{i=1}^{n}\frac{x_i}{\sigma^3} - \frac{2n}{\sigma} + \sum_{i=1}^{n}\frac{2\bar{\alpha}x_i^2\exp\left(\frac{-x_i{}^2}{2\sigma^2}\right)}{\left(1-\bar{\alpha}\exp\left(\frac{-x_i{}^2}{2\sigma^2}\right)\right)\sigma^3}$$

These equations are non linear and can be solved iteratively using the nlm program in R software. For the model selection we use the Akiake Information Criterion (AIC) and the Bayesian Information criterion (BIC) defined as $AIC = -2LogL + 2k$ and $BIC = -2LogL - k\log n$ where k is the number of parameters in the model and n is the sample size.

Asymptotic Confidence Bounds

The normal approximation of the MLEs of $\hat{\alpha}$ and $\hat{\sigma}$ can be used for constructing approximate confidence intervals on the parameters α and σ. The variance covariance matrix I^{-1}, the inverse of the observed information matrix is given by

$$I^{-1} = \begin{bmatrix} \frac{-\partial^2 LogL}{\partial\alpha^2} & \frac{-\partial^2 LogL}{\partial\alpha\,\partial\sigma} \\ \frac{-\partial^2 LogL}{\partial\sigma\,\partial\alpha} & \frac{-\partial^2 LogL}{\partial\sigma^2} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} Var\left(\hat{\alpha}\right) & Cov\left(\hat{\alpha}\,\hat{\sigma}\right) \\ Cov\left(\hat{\sigma}\,\hat{\alpha}\right) & Var\left(\hat{\sigma}\right) \end{bmatrix}$$

By using I^{-1} we can derive the $(1-\delta)\,100\%$ confidence intervals of the parameters α and σ as $\hat{\alpha} \pm Z_{\delta/2}\sqrt{Var\left(\hat{\alpha}\right)}$ and $\hat{\sigma} \pm Z_{\delta/2}\sqrt{Var\left(\hat{\sigma}\right)}$ where $Z_{\delta/2}$ is the upper $\delta/2^{th}$ percentile of the standard normal distribution.

7.1 Fitting Reliability Data

For illustrative study we consider the analysis of two data sets to show how the proposed model works in practice.

Data I

Consider the real data set given in Salman et al (1999) which represents the time between failures (thousand of hours) of secondary reactor pumps. we compare the MOER distribution (4.1) with the Marshall Olkin extended Exponential (MOEE) distribution with parameters (α, σ). The survival function of MOEE distribution is

$$\bar{G}(x) = \frac{\alpha \exp\left(-\frac{x}{\sigma}\right)}{1-\bar{\alpha}\exp\left(-\frac{x}{\sigma}\right)}, \ x \geq 0, \ \alpha > 0, \sigma > 0, \ \bar{\alpha} = 1-\alpha.$$

Table 4 gives the MLEs of the parameters of MOER and MOEE distribution, K-S Statistics, AIC and BIC values and form the obtained values we found that the MOER distribution provides better fit than MOEE distribution asthe MOER distribution has the smallest AIC , BIC values among the data set considered.

Data II

The data represent the observed remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang (2003). Here we estimates the parameters of MOER distribution given in (4.1) and Rayleigh distribution given in (3.1) by the maximum likelihood method. The estimates, AIC, BIC values and the K-S Statistics are given in Table 5. Form the values obtained we can say that the newly proposed MOER distribution performs better for the present data when compared with Rayleigh distribution.

Table 4: Fitting for the MOER and MOEE Distribution

Model	Parameters	Estimates	AIC	BIC	K-S Statistics
	α	0.15001			
MOEE	σ	1.4232	32.53	40.43	0.38714
	α	0.1780			
MOER	σ	1.4235	30.744	32.6367	0.2384

The variance covariance matrix corresponding to the MOER distribution is obtained as follows

$$\begin{bmatrix} 0.0108 & -0.0040 \\ -0.0040 & 0.01051 \end{bmatrix}$$

The 95% confidence intervals for the parameters α and σ are $[0.15682,\ 0.1992]$ and $[1.4029,\ 1.4441]$ respectively.

Table 5: Fitting for the MOER and Rayleigh Distribution

Model	Parameters	Estimates	AIC	BIC	K-S Statistics
Rayleigh	σ	3.2776	951.1810	952.1273	0.3775
	α	1.0500			
MOER	σ	3.4416	837.8750	839.7678	0.3470

The variance covariance matrix corresponding to MOER distribution is obtained as follows

$$\begin{bmatrix} 0.01644 & -0.0007 \\ -0.0007 & 0.00188 \end{bmatrix}$$

The 95% confidence intervals for the parameters α and σ are $[1.0177,\ 1.08223]$ and $[3.4379,\ 3.4452]$ respectively.

8 Conclusion

In this paper we have developed a generalization of Rayleigh distribution namely MOER distribution. We have studied some of the statistical properties of MOER distribution such as probability density function, hazard rate function, compounding property, distribution of order statistics, asymptotic distribution of extreme order statistics etc. The density function of the n^{th} record is obtained and the first and second single moment of n^{th} upper record value is derived. The method of maximum likelihood is derived and we have described two cases of real data application to verify that MOER distribution performs better then the existing models.

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